9

GAME THEORY

9.1. INTRODUCTION

Life is full of struggle and competitions. A great variety of competitive situations is commonly seen in everyday life. For example, candidates fighting an *election* have their conflicting interests, because each candidate is interested to secure more votes than those secured by all others. Besides such pleasurable activities in competitive situations, we come across much more earnest competitive situations, of military battles, advertising and marketing compaigns by competing business firms, etc.

What should be the bid to win a big Government contract in the pace of competition from several contractors ? Game must be thought of, in a broad sense, not as a kind of sport but as competitive situation, a kind of conflict in which somebody must *win* and somebody must *lose*.

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the '***minimax (maximin)*** criterion' of **J. Von Neumann** (called the father of game theory). This criterion implies the assumption of rationality from which it is argued that each player will act so as **to *maximize his minimum gain*** or ***minimize his*** *maximum loss*. The difficulty lies in the deduction from the assumption of 'rationality' that the other player will maximize his minimum gain. There is no agreement even among game theorists that rational players should so act. In fact, rational players do not act apparently in this way, or in any consistent way. Therefore, game theory is generally interpreted as an "as if" theory, that is, as if rational decision maker (player) behaved in some well defined (but arbitrarily selected) way, such as *maximizing the minimum gain*.

The game theory has only been capable of analysing very simple competitive situations. Thus, there has been a great gap between what the theory can handle and most actual competitive situations in industry and elsewhere. So the primary contribution of game theory has been its concepts rather than its formal application to solving real problems.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benefit or satisfaction or suffers loss (negative benefit).

The set of rules defines the ***game***. *Going through the set of rules once by the participants defines a* ***play***.

9.2. CHARACTERISTICS OF GAME THEORY

There can be various types of games. They can be classified on the basis of the following characteristics.

(*i*) ***Chance of strategy*** : If in a game, activities are determined by skill, it is said to be a ***game of strategy;*** if they are determined by chance, it is a *game of chance*. In general, a game may involve game of strategy as well as a game of chance. In this chapter, simplest models of games of strategy will be considered.

(*ii*) ***Number of persons*** : A game is called an n-person game if the number of persons playing is n. The person means an individual or a group aiming at a particular objective.

(*iii*) ***Number of activities*** : These may be *finite* or *infinite*.

(iv) ***Number of alternatives (choices) available to each person*** in a particular activity may also be finite or infinite. A *finite* game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is said to be *infinite*.

(v) ***Information to the players about the past activities of other players*** is completely available, partly available, or not available at all.

(vi) ***Payoff*** **:** A quantitative measure of satisfaction a person gets at the end of each play is called a *payoff*. It is a real-valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \le i \le n$, in an n-person game. If $\Sigma_{i=1}^{n} v_i = 0$, then the game is said to be a *zero-sum game*.

In this chapter, we shall discuss *rectangular games (also called two-person zero-sum)* only.

Q. 1. Write a short note on characteristics of game theory. [Rewa M.Sc. (Math) 93]

2. What is game theory ? List out the assumptions made in the theory of games. [JNTU 2003, 02]

9.3. BASIC DEFINITIONS

1. Competitive Game. A competitive situation is called a ***competitive game*** if it has the following four properties : [JNTU (B. Tech.) 2004, 03; Meerut 2002]

(i) There are finite number (n) of competitors (called players) such that $n \ge 2$. In case $n = 2$, it is called a **two-person game** and in case $n > 2$, it is referred to as an **n-person game.**

(ii) Each player has a list of finite number of possible activities (the list may not be same for each player).

(iii) A play is said to *occur* when each player chooses one of his activities. The choices are assumed to be made simultaneously, *i.e.* no player knows the choice of the other until he has decided on his own.

(iv) Every combination of activities determines an outcome (which may be points, money or any thing else whatsoever) which results in a gain of payments (+ ve, – ve or zero) to each player, provided each player is playing uncompromisingly to get as much as possible. Negative gain implies the loss of same amount.

2. Zero-sum and Non-zero-sum Games. Competitive games are classified according to the number of players involved, *i.e.* as a *two person game, three person game,* etc. Another important distinction is between *zero-sum games* and *nonzero-sum* games. If the players make payments only to each other, *i.e.* the loss of one is the gain of others, and nothing comes from outside, the competitive game is said to be *zero-sum.*

Mathematically, suppose an n-person game is played by n players $P_1, P_2, \ldots, P_n$ whose respective pay-offs at the end of a play of the game are $v_1, v_2, \ldots v_n$ then, the game will be called zero-sum if $\sum_{i=1}^{n} v_i = 0$ at each play of the game. [JNTU (Mech. & Prod.) 2004]

A game which is not zero-sum is called a *nonzero-sum game.* Most of the competitive games are zero-sum games. An example of a nonzero-sum game is the 'poker' game in which a certain part of the pot is removed from the 'house' before the final payoff.

3. Strategy. A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt. In other words, a strategy for a given player is a set of rules (programmes) that specifies which of the available course of action he should make at each play. This strategy may be of two kinds : [JNTU (B. Tech.) 2004, 03; Meerut 2002; IGNOU 2001, 2000, 98, 97]

(i) ***Pure Strategy.*** : If a player knows exactly what the other player is going to do, a *deterministic* situation is obtained and objective function is to maximize the gain. *Therefore, the pure strategy is a decision rule always to select a particular course of action.* [Meerut 2002]

A pure strategy is usually represented by a number with which the course of action is associated.

(ii) ***Mixed Strategy.*** [Agra 92; Kerala (Stat.) 83] : If a player is guessing as to which activity is to be selected by the other on any particular occasion, a *probabilistic* situation is obtained and objective function is to maximize the *expected gain.* [Meerut 2003, 02]

Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Mathematically, a mixed strategy for a player with m (≥ 2) possible courses of action, is denoted by the set S of m non-negative real numbers whose sum is unity, representing probabilities with which each course of action is chosen. If x_i ($i = 1, 2, 3, \ldots, m$) is the probability of choosing the course i, *then*

$$S = (x_1 , x_2 , x_3 , \ldots , x_m) \qquad \ldots(9 \cdot 1)$$

subject to the conditions

$$x_1 + x_2 + x_3 + \ldots + x_m = 1 \qquad \ldots(9 \cdot 2)$$

and

$$x_1 \geq 0 , x_2 \geq 0 , x_3 \geq 0 , \ldots , x_m \geq 0 . \qquad \ldots(9 \cdot 3)$$

Note. It should be noted that if some $x_i = 1$, ($i = 1, 2, \ldots, m$) and all others are zero, the player is said to use a pure strategy. Thus, the pure strategy is a particular case of mixed strategy.

4. Two-Person, Zero-Sum (*Rectangular*) Games. A game with only two players (say, *Player A* and *Player B*) is called a *'two-person, zero-sum game'* if the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero. **[JNTU (B. Tech.) 2003]**

Two-person, zero-sum games, are also called *rectangular games* as these are usually represented by a payoff matrix in rectangular form. **[Meerut (OR) 2003]**

5. Payoff Matrix. Suppose the player A has m activities and the player B has n activities. Then a payoff matrix can be formed by adopting the following rules : **[Meerut (OR) 2003; JNTU (B. Tech) 97]**

(*i*) Row designations for each matrix are activities available to player A.

(*ii*) Column designations for each matrix are activities available to player B.

(*iii*) Cell entry 'v_{ij}' is the payment to player A in A's payoff matrix when A chooses the activity i and B chooses the activity j.

(*iv*) With a 'zero-sum, two person game', the cell entry in the player B's payoff matrix will be negative of the corresponding cell entry 'v_{ij}' in the player A's payoff matrix so that sum of payoff matrices for player A and player B is ultimately zero.

Table 9·1. The player A's payoff matrix

		Player B					
		1	2	...	j	...	n
	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player A	:	:	:		:		:
	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:		:		:
	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

Table 9·2. The player B's payoff matrix

		Player B					
		1	2	...	j	...	n
	1	$-v_{11}$	$-v_{12}$	...	$-v_{1j}$	...	$-v_{1n}$
	2	$-v_{21}$	$-v_{22}$	...	$-v_{2j}$	...	$-v_{2n}$
Player A	:	:	:		:		:
	i	$-v_{i1}$	$-v_{i2}$	...	$-v_{ij}$	...	$-v_{in}$
	:	:	:		:		:
	m	$-v_{m1}$	$-v_{m2}$	...	$-v_{mj}$	...	$-v_{mn}$

Note. Further, there is no need to write the B's payoff matrix as it is just the –ve of A's payoff matrix in a *zero-sum two-person game*. Thus, if 'v_{ij}' is the gain to A, then '$-v_{ij}$' will be the gain to B.

In order to make the above concepts a clear, consider the coin matching game involving two players only. Each player selects either a head H or a tail T. If the outcomes match (H, H or T, T), A wins Re 1 from B, otherwise B wins Re 1 from A. This game is a two-person zero-sum game, since the winning of one player is taken as losses for the other. Each has his choices between two *pure* strategies (H or T). This yields the following (2 × 2) payoff matrix to player A.

Table 19.3.

		B	
		H	T
A	H	+1	–1
	T	–1	+1

It will be shown later that the optimal solution to such games requires each player to play one pure strategy or a mixture of pure strategies.

Q. 1. State the four properties which a competitive situation should have, if it is to be called a competitive game. **[IGNOU (B.Com) 91]**

2. What is the problem studied in game theory ?

3. Define :
   (i) Competitive game
   (ii) Pure strategies
   (iii) Mixed strategies

(iv) Two-person, Zero-sum (or Rectangular games) [Agra 94]
(v) Payoff matrix [IGNOU 2001, 2000, 98, 97; JNTU (B. Tech) 97]

4. (i) Explain zero-sum two-person game giving suitable example.
(ii) What is a zero-sum two-person game? [IGNOU 2001, 2000, 98, 97; Agra 92]
(iii) Explain the difference between pure strategy and mixed strategy.

9.4. MINIMAX (MAXIMIN) CRITERION AND OPTIMAL STRATEGY

The 'minimax criterion of optimality' states that if a player lists the worst possible outcomes of all his potential strategies, he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Example 1. *Consider (two-person, zero-sum) game matrix which represents payoff to the player A. Find the optimal strategy, if any.* [*See Table 9·4*]

Table 9·4

			B			
	Saddle Point	I	II	III	Row minimum	
	I	-3	-2	$\boxed{-3}$	-3	
A	II	2	$\boxed{0}$	2	$\boxed{0}$	Maximin Value ($\underline{v}$)
	III	$\boxed{5}$	-2	$\boxed{-4}$	-4	
Column maximum		$\boxed{5}$	$\boxed{0}$	$\boxed{6}$		
			Minimax Value ($\bar{v}$)			

Solution. The player A wishes to obtain the largest possible 'v_{ij}' by choosing one of his activities (I, II, III), while the player B is determined to make A's gain the minimum possible by choice of activities from his list (I, II, III). The player A is called the ***maximizing player*** and B the ***minimizing player.***

If the player A chooses the Ist activity, then it could happen that the player B also chooses his Ist activity. In this case the player B can guarantee a gain of at least -3 to player A, *i.e.*

$$\min\{-3, -2, 6\} = -3$$

Similarly, for other choices of the player A, *i.e.* II and III activities, B can force the player A to get only 0 and -4, respectively, by his proper choices from (I, II, III), *i.e.*

$$\min\{2, 0, 2\} = 0 \quad \text{and} \quad \min\{5, -2, -4\} = -4$$

The minimum value in each row guaranteed by the player A is indicated by 'row minimum' in (*Table 9·4*). The best choice for the player A is to maximize his least gains -3, 0, -4 and opt II strategy which assures at most the gain 0, *i.e.*

$$\max\{-3, 0, -4\} = \boxed{0}$$

In general, the player A should try to maximize his least gains or to find out "$\max_i \min_j v_{ij}$"

Player B, on the other hand, can argue similarly to keep A's gain the minimum. He realizes that if he plays his Ist pure strategy, he can loose no more than $5 = \max\{-3, 2, 5\}$ regardless of A's selections. Similar arguments can be applied for remaining strategies II and III. Corresponding results are indicated in *Table 9·4* by '*column maximum*'. The player B will then select the strategy that minimizes his maximum losses. This is given by the strategy II and his corresponding loss is given by

$$\min\left\{\boxed{5}, \boxed{0}, \boxed{6}\right\} = \boxed{0}$$

The player A's selection is called the ***maximin strategy*** and his corresponding gain is called the ***maximin value*** or ***lower value*** ($\underline{v}$) of the game. The player B's selection is called the ***minimax value*** or ***upper value*** ($\bar{v}$) of the game. The selections made by player A and B are based on the so called ***minimax*** (or ***maximin***) ***criterion***. It is seen from the governing conditions that the minimax (upper) value $\bar{v}$ is greater than or equal to the maximin (lower) value $\underline{v}$ (see *Theorem 9·1*). In the case where equality holds *i.e.*,

$$\max_i \min_j v_{ij} = \min_j \max_i v_{ij} \quad \text{or} \quad \underline{v} = \bar{v}, \qquad \ldots(9\cdot4)$$

the corresponding pure strategies are called the *'optimal'* strategies and the game is said to have a ***saddle point.*** It may not always happen as shown in the following example.

Note. For convenience, the minimum values are shown by 'O' and maximum values by '☐' in Table 9·4.

Q. 1. What is a competitive situation called a game ? What is the maximin criterion of optimality ?

2. What is a game in the game theory ? What are the properties of a game ? Explain the best (optimal) strategy on the basis of minimax criterion of optimality.

3. What are the assumptions made in the theory of games?

4. Explain Maxi-Min and Mini-Max principle used in Game Theory. **[JNTU (B. Tech) 2000; Agra 94]**

Example 2. *Consider the following game :*

		B		
	$i \backslash j$	1	2	3
	1	3	−4	8
A	2	−8	5	−6
	3	6	−7	6

As discussed in ***Example 1***, $\max_i \min_j v_{ij} = 4, \quad \min_j \max_i v_{ij} = 5.$

Also, $\max_i \min_j v_{ij} < \min_j \max_i v_{ij}$

Such games are said to be the games ***without saddle point.***

Example 3. *Find the range of values of p and q which will render the entry (2, 2) a saddle point for the game :* **[JNTU (B. Tech.) 2003]**

	Player B		
Player A	2	4	5
	10	7	q
	4	p	6

Solution. First ignoring the values of p and q determine the maximin and minimax values of the payoff matrix as below :

		Player B			
		B_1	B_2	B_3	Row Min..
	A_1	2	4	5	2
Player A	A_2	$\boxed{10}$	$\boxed{7}$	q	7
	A_3	4	p	$\boxed{6}$	4
	Column Max.	10	7	6	

Since the entry (2, 2) is a saddle point, maximin value $\underline{v} = 7$, minimax value $\bar{v} = 7$.

This imposes the condition on p as $p \le 7$ and on q as $q \ge 7$. Hence the range of p and q will be $p \le 7$, $q \ge 7$.

Theorem 9·1. *Let* $\{v_{ij}\}$ *be the payoff matrix for a two-person zero-sum game. If* $\underline{v}$ *denotes the maximin value and* $\bar{v}$ *the minimax value of the game, then* $\bar{v} \ge \underline{v}$. *That is,* $\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}]$ **[Meerut (Stat.) 90]**

Proof. We have, $\max_i \{v_{ij}\} \ge v_{ij}$ for any j, and

$\min_i \{v_{ij}\} \le v_{ij}$ for any i .

Let the above maximum be attained at $i = i^*$ and the minimum be attained at $j = j^*$. So

$$v_{i^*j} \ge v_{ij} \ge v_{ij^*} \qquad \text{for any } i \text{ and } j.$$

This implies that

$$\min_j \{v_{i^*j}\} \ge v_{ij} \ge \max_i \{v_{ij^*}\} \qquad \text{for any } i \text{ and } j\,.$$

Hence

$$\min_j [\max_i \{v_{ij}\}] \ge \max_i [\min_j \{v_{ij}\}] \quad \text{or} \quad \bar{v} \ge \underline{v}\,.$$

9·5. SADDLE POINT, OPTIMAL STRATEGIES AND VALUE OF THE GAME

Definitions :

Saddle Point. *A **saddle point** of a payoff matrix is the position of such an element in the payoff matrix which is minimum in its row and maximum in its column.* **[JNTU (MCA III) 2004, 97; Meerut 2003, 02; IGNOU 99, 98]**

Mathematically, *if a payoff matrix* $\{v_{ij}\}$ *is such that* $\max_i [\min_j \{v_{ij}\}] = \min_j [\max_i \{v_{ij}\}] = v_{rs}$ (say), *then the matrix is said to have a saddle point* (r, s).

Optimal Strategies. *If the payoff matrix* $\{v_{ij}\}$ *has the saddle point* (r, s), *then the players (A and B) are said to have* ***rth and sth optimal strategies****, respectively.* **[Meerut (OR) 2003; JNTU 97]**

3. Value of Game. *The payoff* (v_{rs}) *at the saddle point* (r, s) *is called the* ***value of game*** *and it is obviously equal to the maximin* ($\underline{v}$) *and minimax value* ($\overline{v}$) *of the game.* **[IGNOU 99, 98]**

A game is said to be a ***fair game*** if $\overline{v} = \underline{v} = 0$. A game is said to be ***strictly determinable*** if $\overline{v} = v = \underline{v}$.

Note. A saddle point of a payoff matrix is, sometimes, called the equilibrium point of the payoff matrix.

In *Example 1*, $\underline{v} = \overline{v} = 0$. This implies that the game has a saddle point given by the entry (2, 2) of payoff matrix. The value of the game is thus equal to zero and both players select their strategy as the optimal strategy. In this example, it is also seen that no player can improve his position by other strategy.

In general, a matrix need not have a saddle point as defined above. Thus, these definitions of optimal strategy and value of the game are not adequate to cover all cases so need to be generalized. The definition of a saddle point of a function of several variables and some theorems connected with it form the basis of such generalization. These theorems are presented in *Sec. 9·24.*

Rules for Determining a Saddle Point :

1. ***Select the minimum element of each row of the payoff matrix and mark them by 'O'.***
2. ***Select the greatest element of each column of the payoff matrix and mark them by '□'.***
3. ***If there appears an element in the payoff matrix marked by 'O' and '□' both, the position of that element is a saddle point of the payoff matrix.***

Q. 1. Define : (i) Competitive Game, (ii) Payoff matrix, (iii) Pure and mixed strategies, (iv) Saddle point, (v) Optimal strategies, and (vi) Rectangular game **[JNTU 2000; Kanpur M.Sc. (Maths.) 93]**

2. Explain "best strategy" on the basis of minimax criterion of optimalities.

3. Describe the maximin principle of game theory. What do you understand by pure strategies and saddle point. **[SJMIT (BE Mech.) 2002; Punjabi (M.B.A.) 90]**

4. Define saddle point and the value of game with examples. **[Meerut 2002; GNDU (B. Com.) 91]**

5. Define saddle point. Is it necessary that a said game should always possess a saddle point ?

6. State the rules for detecting a saddle point.

7. What is 'strictly determined game' ? When a game is said to be determinable ?

8. Write short notes on the following : (i) Pure Strategy, (ii) Mixed Strategy, (iii) Max-Min Criterion.

9. Let $A = \{a_{ij}\}$ be an $m \times n$ payoff matrix for a zero-sum two-person game. Define a Saddle point for matrix A and show that the value of the game is equal to the saddle value.

10. Differentiate between strictly determinable game and non-determinable games. **[JNTU (Mech. & Prod.) 2004]**

EXAMINATION PROBLEMS

1. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of strictly determinable games : **[JNTU (B. Tech.) 2003]**

(i) Player B

$$\text{Player } A \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}$$

[Ans. Not fair, $v = 1$]

(ii) Player B

$$\text{Player } A \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix}$$

[Ans. Not fair, (I, II), $v = -5$]

2. Consider the game G with the following payoff matrix :

Player B

$$\text{Player } A \begin{bmatrix} 2 & 6 \\ -2 & \mu \end{bmatrix}$$

(i) Show that G is strictly determinable whatever μ may be. (ii) Determine the value of G.

[Ans. (I, I), $v = 2$] **[Jodhpur M.Sc. (Math.) 92]**

3. Find out whether there is any saddle point in the following problem :

Player B

$$\text{Player } A \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$

[Ans. Saddle point does not exist.]

4. For the game with payoff matrix :

Player B

$$\text{Player } A \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix}$$

determine the best strategies for players A and B and also the values of the game for them. Is this game (*i*) fair (*ii*) strictly determinable ?

[Ans. (I, II), $v = -2$.
Game is strictly determinable and not fair.]

9.6. SOLUTION OF GAMES WITH SADDLE POINT

To obtain a solution of a rectangular game, it is feasible to find out :

(i) the best strategy for player A *(ii)* the best strategy for player B, and *(iii)* the value of the game (v_{rs}).

It is already seen that the best strategies for players A and B will be those which correspond to the row and column, respectively, through the saddle point. The value of the game to the player A is the element at the saddle point, and the value to the player B will be its negative.

9.7. ILLUSTRATIVE EXAMPLES

Example 4. *Player A can choose his strategies from* $\{A_1, A_2, A_3\}$ *only. while B can choose from the set* (B_1, B_2) *only. The rules of the game state that the payments should be made in accordance with the selection of strategies :*

Strategy Pair Selected	*Payments to be Made*	*Strategy Pair Selected*	**Payments to be Made**
(A_1, B_1)	*Payer A Pays Re. 1 to player B*	(A_2, B_2)	*Player B pays Rs 4 to player A*
(A_1, B_2)	*Player B pays Rs. 6 to player A*	(A_3, B_1)	*Player A pays Rs 2 to player B*
(A_2, B_1)	*Player B pays Rs 2 to player A*	(A_3, B_2)	*Player A pays Rs. 6 to player B*

What strategies should A and B play in order to get the optimum benefit of the play ?

Solution. With the help of above rules the following payoff matrix is constructed :

$$\begin{array}{ccc|c|c|} & & \multicolumn{2}{c}{\text{Player } B} \\ & & B_1 & B_2 \\ \hline & A_1 & \textcircled{-1} & \boxed{6} \\ \hline \text{Player } A & A_2 & \boxed{\textcircled{2}} & 4 \\ \hline & A_3 & -2 & \textcircled{-6} \\ \hline \end{array}$$

The payoffs marked 'O' represent the minimum payoff in each row and those marked '☐' represent the maximum payoff in each column of the payoff matrix.

Obviously, the matrix has a saddle point at position (2, 1) and the value of the game is 2.

Thus, the optimum solution to the game is given by :

(i) the optimum strategy for player A is A_2 ; *(ii)* the optimum strategy for player B is B_1 ; and

(iii) the value of the game is Rs. 2 for player A and Rs. (– 2) for player B.

Also, since $v \neq 0$, the game is not fair, although it is strictly determinable.

Example 5. *The payoff matrix of a game is given. Find the solution of the game to the player A and B.*

$$\begin{array}{cc} & B \\ & \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

[JNTU (MCA III) 2004, (B. Tech.) 2000, 99]

Solution. First find out the saddle point by encircling each row minima and putting squares around each column maxima.

The saddle point thus obtained is shown by having a circle and square both (*Table 9·5*)

Table 9.5

Optimum Strategy for B ↑

	I	II	III	IV	V	Row Minimum
I	$\textcircled{-2}$	0	0	$\boxed{5}$	3	$\textcircled{-2}$
Optimum Strategy for A ← II	3	2	$\boxed{\textcircled{1}}$	2	2	$\boxed{\textcircled{1}}$ ····· Maximin Value ($\underline{v}$)
III	$\textcircled{-4}$	-3	0	-2	$\boxed{6}$	-4
IV	$\boxed{5}$	$\boxed{3}$	-4	2	$\textcircled{-6}$	$\textcircled{-6}$
Column Maximum	$\boxed{5}$	$\boxed{3}$	$\boxed{\textcircled{1}}$	$\boxed{5}$	$\boxed{6}$	

Minimax Value ($\bar{v}$)

Hence, the solution to this game is given by, *(i)* the best strategy for player A is 2nd; *(ii)* the best strategy for player B is 3rd; and *(iii)* the value of the game is 1 to player A and – 1 to player B.

Example 6. *Solve the game whose payoff matrix is given by*

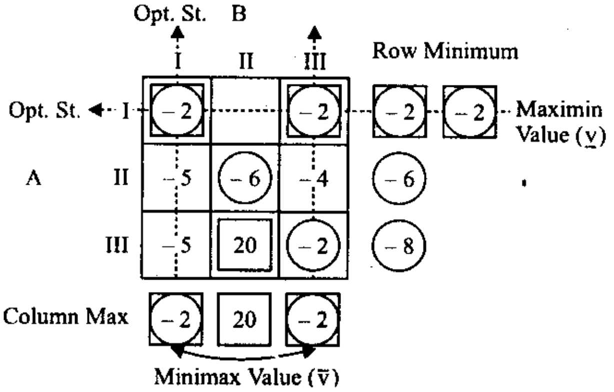
$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} -2 & 15 & -2 \\ -5 & -6 & -4 \\ -5 & 20 & -8 \end{bmatrix} \end{array}$$

[Kanpur M.Sc. (Math.) 96; Rewa (M.P.) 93]

Solution. *Table* 9·6 may be formed as explained earlier.

This game has two saddle points in positions (1, 1) and (1, 3). Thus, the solution to this game is given by,

(i) the best strategy for the player A is I, *(ii)* the best strategy for the player B is either I or III, *i.e.* the player B can use either of the two strategies (I, III), and *(iii)* the value of the game is – 2 for player A and + 2 for player B.



EXAMINATION PROBLEMS

Find the saddle point (or points) and hence solve the following games :

1. **Player B**

$$\textbf{Player A}\ \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix} \end{array}$$

[Ans. (A_2, B_2), $v = 5$]

2. B

$$A\ \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \end{array}$$

[Ans. (A_2, B_2), $v = 4$]

3. B

$$A\ \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \begin{bmatrix} -5 & 2 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ 4 & -2 & 0 & -5 \end{bmatrix} \end{array}$$

[Ans. (II, III), $v = 4$]

4. B

$$A\ \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array}\begin{array}{c} \begin{array}{ccccc} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{array} \\ \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \end{array}$$

[Ans. (II, III), $v = 4$]

5.

$$\begin{array}{c} \\ \text{I} \\ \text{II} \end{array}\begin{array}{c} \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

[Ans. (I, I), (I, III), $v = 6$]

6.

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array}\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -3 \\ 2 & 3 & 1 \\ -4 & 2 & -1 \end{bmatrix} \end{array}$$

[Ans. (R_2, C_3), $v = 1$]

7. Solve the game whose payoff matrix is given by

Player B

$$\text{Player } A \begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

[Agra 98]

[Ans. (I, I) or (I, III), $v = 1$ for A, $v = -1$ for B]

8. For the following payoff matrix for firm A, determine the optimal strategies for both the firms and the value of the game (using maximin-minimax principle) :

Firm B

$$\text{Firm } A \begin{bmatrix} 3 & -1 & 4 & 6 & 7 \\ -1 & 8 & 2 & 4 & 12 \\ 16 & 8 & 6 & 14 & 12 \\ 1 & 11 & -4 & 2 & 1 \end{bmatrix}$$

[Ans. (III, III), $v = 6$ for A]

9. Solve the games whose payoff matrices are given below :

(a) Player B

$$\text{Player A}\ \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} -3 & -1 & 6 \\ 2 & 0 & 2 \\ 5 & -2 & -4 \end{bmatrix} \end{array}$$

(b) Player B

$$\text{Payer } A \begin{bmatrix} 15 & 2 & 3 \\ 6 & 5 & 7 \\ -7 & 4 & 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} -5 & 5 & 0 & 7 \\ 2 & 6 & 1 & 8 \\ -4 & 0 & 1 & -3 \end{bmatrix}$$

[Ans. (a) (A_2, B_2), $v = 0$; (b) (A_2, B_2), $v = 5$; (c) (A_2, A_3), $v = 1$] **[Kanpur 2000]**

10. For what values of 'a' the game with the following pay-off matrix is strictly determinable ?

[JNTU (Mech. & Prod.) 2004]

$$\begin{array}{cc|ccc|} & & & B & \\ & & B_1 & B_2 & B_3 \\ \hline & A_1 & a & 6 & 2 \\ A & A_2 & -1 & a & -7 \\ & A_3 & -2 & 4 & a \\ \hline \end{array}$$

9.8. RECTANGULAR GAMES WITHOUT SADDLE POINT

As discussed earlier, if the payoff matrix $\{v_{ij}\}$ has a saddle point (r, s), then $i = r, j = s$ are the optimal strategies of the game and the payoff v_{rs} $(= v)$ is the value of the game. On the other hand, if the given matrix has no saddle point, the game has no optimal strategies. The concept of optimal strategies can be extended to all matrix games by introducing a probability with choice and mathematical expectation with payoff.

Let player A choose a particular activity i such that $1 \le i \le m$ with probability x_i. This can also be interpreted as the relative frequency with which A chooses activity i from number of activities of the game. Then set $\mathbf{x} = \{x_i, 1 \le i \le m\}$ of probabilities constitute the strategy of A. Similarly, $\mathbf{y} = \{y_j, 1 \le j \le n\}$ defines the strategy of the player B.

Thus, the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of non-negative numbers satisfying $x_1 + x_2 + \ldots + x_m = 1$ is called the *mixed strategy of the player A*. Similarly, the vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ of non-negative numbers satisfying $y_1 + y_2 + \ldots + y_n = 1$ is called the *mixed strategy of the player B*.

Consider the symbol S_m which denotes the set of ordered m-tuples of non-negative numbers whose sum is unity and $\mathbf{x} \in S_m$. Similarly, $\mathbf{y} \in S_n$. Unless otherwise stated, assume that $\mathbf{x} \in S_m$ and $\mathbf{y} \in S_n$, where $\mathbf{x}$ and $\mathbf{y}$ are mixed strategies of player A and B, respectively.

The mathematical expectation of the payoff function $E(\mathbf{x}, \mathbf{y})$ in a game whose payoff matrix is $\{v_{ij}\}$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i \, v_{ij}) \, y_j \qquad = \mathbf{x}^T \mathbf{v} \, \mathbf{y} \quad (\textit{in matrix form})$$

where **x** *and* **y** *are the mixed strategies of players A and B*, respectively,

Thus the player A should choose **x** so as to maximize his minimum expectation and the player B should choose **y** so as to minimize the player A's greatest expectation. In other words, the player

A tries for $\max_{\mathbf{x}} \min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$ and B tries for $\min_{\mathbf{y}} \max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y})$.

At this stage it is possible to define the *strategic saddle point* of the game with mixed strategies.

Strategic Saddle Point. ***Definition.*** *If* $\min_{\mathbf{y}} \max_{\mathbf{x}} E(\mathbf{x}, \mathbf{y}) = E(\mathbf{x}_0, \mathbf{y}_0) = \max_{\mathbf{x}} \min_{\mathbf{y}} E(\mathbf{x}, \mathbf{y})$, *then* $(\mathbf{x}_0, \mathbf{y}_0)$ *is called the* ***strategic saddle point*** *of the game where* $\mathbf{x}_0$ *and* $\mathbf{y}_0$ *define the optimal strategies, and* $v = E(\mathbf{x}_0, \mathbf{y}_0)$ *is the value of the game.*

According to the *minimax theorem* (***Section 9·11***), a strategic saddle point will always exist.

Example 7. *In a game of matching coins with two players, suppose one player wins Rs. 2 when there are two heads and wins nothing when there are two tails; and losses Re. 1 when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.*

Solution. The payoff matrix (for the player A) is given by

$$\begin{array}{cc|cc|l} & & \text{Player } B & & \\ & & H & T & \text{Row Min.} \\ \hline & H & 2 & -1 & -1 \\ \text{Player } A & T & -1 & 0 & -1 \quad \text{Maximin (lower) value } (\underline{v}) \\ \hline & \text{Column Max :} & 2 & 0 & \\ & & \uparrow & & \\ & & \text{Minimax (upper) value } (\bar{v}) & & \end{array}$$

Here, maximin value $(\underline{v}) = -1 \ne$ minimax value $(\bar{v}) = 2$.

So the matrix is whithout saddle point.

Now, let us outline here how one finds the best strategies for such games and the expected amounts to be gained or lost by the players.

Let the player A plays H with probability x and T with probability $1 - x$ so that $x + (1 - x) = 1$. Then, if the player B plays H all the time, A's expected gain will be

$$E(A, H) = x.2 + (1 - x)(-1) = 3x - 1. \qquad \ldots(9{\cdot}6)$$

Similarly, if the player B plays T all the time, A's expected gain will be

$$E(A, T) = x(-1) + (1-x)\,0 = -x. \qquad \text{...(9·7)}$$

It can be shown mathematically that if the player A chooses x such that

$$E(A, H) = E(A, T) = E(A), \text{ say}, \qquad \text{...(9·8)}$$

then this will determine the best strategy for him.

Thus,
$$3x - 1 = -x \text{ or } x = 1/4 \qquad \text{...(9·9)}$$

Therefore, best strategy for the player A is to play H and T with probability 1/4 and 3/4, respectively. Since this is a mixed strategy, it is usually denoted by the set {1/4, 3/4}. So expected gain for the player A is given by

$$E(A) = \tfrac{1}{4}\,.\,2 + \tfrac{3}{4}(-1) = -\tfrac{1}{4}$$

Now, whatever be the set $\{y, 1-y\}$ of probabilities with which the player B plays either H or T, A's expected gain will always remain equal to $-1/4$. To verify this,

$$E(A, y, 1-y) = y\left[\tfrac{1}{4}\,.\,2 + \tfrac{3}{4}(-1)\right] + (1-y)\left[\tfrac{1}{4}(-1) + \tfrac{3}{4}\,0\right]$$

$$= y\left(-\tfrac{1}{4}\right) + (1-y)\left(-\tfrac{1}{4}\right) = -\tfrac{1}{4}. \qquad \text{...(9·10)}$$

The same procedure can be applied for the player B. Let the probability of the choice of H be denoted by y and that of T by $(1-y)$. For best strategy of the player B,

$$E(B, H) = E(B, T) = E(B), \text{ say} \qquad \text{...(9·11)}$$

or
$$y\,.\,2 + (1-y)(-1) = y(-1) + (1-y)\,0$$

or
$$4y = 1$$

or
$$y = 1/4 \quad \text{and} \quad \text{therefore} \quad 1 - y = 3/4.$$

Therfore, $E(B) = \frac{1}{4.}\,2 + \frac{3}{4}(-1) = -\frac{1}{4}$.

Here, $E(A) = E(B) = -1/4$. Thus, the complete solution of the game is :

(i) The player A should play H and T with probabilities 1/4 and 3/4, respectively. Thus, A's optimal strategy is $\mathbf{x}_0 = (1/4, 3/4)$.

(ii) The player B should play H and T with probabilities 1/4 and 3/4, respectively. Thus, B's optimal strategy is $\mathbf{y}_0 = (1/4, 3/4)$.

(iii) The expected value of the game is $-1/4$ to the player A. Here $(\mathbf{x}_0, \mathbf{y}_0)$ is the strategic saddle point of this game.

Remark. Although this example can be easily solved by using the formula of **Section** 9·13, the present discussion will be of great help in understanding the further discussion.

EXAMINAITON PROBLEMS

1. Find the optimal strategies for the games for which the pay off matrices are given below. Also, find the value of the game.

(a)
$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ P_1 \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \end{array}$$

[**Ans.** (1/2, 1/2), (1/4, 3/4); $v = 5/2$]

(b)
$$\begin{array}{cc} & P_2 \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ P_1 \begin{array}{c} \text{I} \\ \text{II} \end{array} & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$$

[**Ans.** (1/3, 2/3), (3/5, 2/5); $v = 0$]

2. For the game with the following payoff matrix for the row player, determine the optimal strategies for both the players and the value of the game :

(a) $\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}$

[**Ans.** (1/4, 3/4), (1/4, 3/4); $v = -3/4$]

(b) $\begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix}$

[**Ans.** (2/5, 3/5), (1/2, 1/2) $v = 4$]

1\. A game has the payoff matrix $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$. Show that $E(x, y) = 1 - 2x\left(y - \frac{1}{2}\right)$ and deduce that in the solution of the game the first player follows a pure strategy while the second has infinite number of mixed strategies. **[Raj. (M.Phil.) 92]**

4. State the fundamental theorem of rectangular games. Show that $\max\limits_{i} \min\limits_{j} a_{ij} \le \min\limits_{j} \max\limits_{i} a_{ij}$ in the arbitrary matrix :

$$\begin{pmatrix} a_{11} & a_{12} \cdots\cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots\cdots a_{2n} \\ \vdots & \vdots \qquad\quad \vdots \\ a_{m1} & a_{m2} \cdots\cdots\cdots a_{mn} \end{pmatrix}$$

9.9. MINIMAX-MAXIMIN PRINCIPLE FOR MIXED STRATEGY GAMES

It has been observed earlier that if a game does not have a saddle point, two players cannot use the maximin-minimax (pure) strategies as their optimal strategies. This failure of the minimax-maximum (pure) strategies, in general, give an optimal solution to the game and led to the idea of using mixed strategies. Each player, instead of selecting pure strategies only, may play all his strategies according to a predetermined set of probabilities.

Let $x_1, x_2, \dots, x_m$ and $y_1, y_2, y_3, \dots y_n$ be the probabilities of two players A and B, respectively to select their pure strategies.

Then,
$$x_1 + x_2 + x_3 + \dots + x_m = 1 \qquad \dots(9\cdot12)$$
and
$$y_1 + y_2 + y_3 + \dots + y_n = 1, \qquad \dots(9\cdot13)$$
where $x_i \geq 0$ and $y_j \geq 0$ for all i and j. Thus if v_{ij} represents the (i, j)th entry of the game matrix, probabilities x_i and y_j will appear (*Table 9·7*).

Table 9·7

Player B

	Probabilties→ ↓	j / i	y_1	y_2	...	y_j	...	y_n
		i	1	2	...	j	...	n
	x_1	1	v_{11}	v_{12}	...	v_{1j}	...	v_{1n}
	x_2	2	v_{21}	v_{22}	...	v_{2j}	...	v_{2n}
Player A	:	:	:	:		:		:
	x_i	i	v_{i1}	v_{i2}	...	v_{ij}	...	v_{in}
	:	:	:	:		:		:
	x_m	m	v_{m1}	v_{m2}	...	v_{mj}	...	v_{mn}

The solution of mixed strategy problem is also based on the minimax criterion given in ***Section 9·4***. The only difference is that the player A selects probabilities x_i which *maximize his minimum 'expected' gain* in a column, while the player B selects the probabilities which *minimize his maximum 'expected' loss* in a row.

Mathematically, the minimax criterion for a mixed strategy is as follows :

The player A selects x_i $(x_i \geq 0, \sum_{i=1}^{m} x_i = 1)$ which gives the lower value of the game

$$\max_{x_1, x_2, \dots, x_m} [\min \{(v_{11}x_1 + v_{21}x_2 + \dots + v_{m1}x_m), (v_{12}x_1 + v_{22}x_2 + \dots + v_{m2}x_m), \dots, (v_{1n}x_1 + v_{2n}x_2 + \dots + v_{mn}x_m)\}] \qquad \dots(9\cdot14a)$$

or more precisely,

$$\underline{v} = \max_{x_i} \left[\min \left\{ \sum_{i=1}^{m} v_{i1}\, x_i, \sum_{i=1}^{m} v_{i2} x_i, \dots, \sum_{i=1}^{m} v_{in}\, x_i \right\} \right] \qquad \dots(9\cdot14b)$$

Similarly, the player B chooses y_j $(y_j \geq 0, \sum_{j=1}^{n} y_j = 1)$ which gives the upper value of the game

$$\overline{v} = \min_{y_j} \left[\max \left\{ \sum_{j=1}^{n} v_{1j}\, y_j, \sum_{j=1}^{n} v_{2j}\, y_j, \dots, \sum_{j=1}^{n} v_{mj}\, y_j \right\} \right] \qquad \dots(9\cdot15)$$

These values are referred to the *maximin* ($\underline{v}$) and the *minimax* ($\overline{v}$) expected values, respectively.

In pure strategies, the relationship, $\overline{v} \geq \underline{v}$, holds in general. When x_i and y_j correspond to the optimal solution, this relation holds in 'equality' sense and the 'expected' values thus obtained become equal to the (optimal) expected values of the game. This result follows from the *minimax theorem* (called the *fundamental theorem of rectangular games*) which is derived in ***Sec. 9·12***.

We shall require the following *Lemma* in ***Sec 9·10***.

Lemma. *Let* A = (v_{ij}) *be the payoff matrix of an m × n game. If* B = (v'_{ij}) *is obtained from A by adding a constant c to every element of A, then an optimal strategy for B is also an optimal strategy for A.*

Proof. Let v' be the value of the game with payoff matrix B. Then for the strategies **x, y,**

$$\sum_{i=1}^{m} \sum_{j=1}^{n} v'_{ij} x_i y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} v_{ij} x_i y_j + c .$$

If **x***, **y*** are optimal strategies for the game **B**, then

$$\sum_i \sum_j v'_{ij} x_i y_j^* \le v' \le \sum_i \sum_j v'_{ij} x_i^* y_j \Rightarrow \sum_i \sum_j v_{ij} x_i y_j^* + c \le v' \le \sum_i \sum_j v_{ij} x_i^* y_j + c$$

$$\therefore \quad \sum_i \sum_j v_{ij} x_i y_j^* \le v' - c \le \sum_i \sum_j v_{ij} x_i^* y_j .$$

Thus $\mathbf{x}^*$, $\mathbf{y}^*$ are optimal for game A with the value of game $v = v' - c$.

Hence arbitrarily chosen constant c can be added to each element of A and then we can solve the resulting game B. The value v of the original game is then obtained simply by subtracting the constant c from the value of the game B. Constant c is chosen so large that $v_{ij} + c$ is positive (> 0) for all i and j, so that the value of the game is certainly positive.

Q. 1. Define the terms "maximin element, minimax element and saddle point" of the payoff matrix of a two-person zero-sum games.

2. Explain 'minimax criterion' as applied to the theory of games. **[Bhubneshwar (IT) 2004]**

3. Let (v_{ij}) be the payoff matrix for a two-person zero-sum game. If $\underline{v}$ denotes the maximin value and $\bar{v}$ the minimax value of the game, then prove that $\bar{v} \ge \underline{v}$. That is, $\min_j [\max_i (v_{ij})] \ge \max_i [\min_j (v_{ij})]$. **[Meerut (Stat.) 90]**

EXAMINATION PROBLEMS

Find the minimax and maximin value of the following games :

(i) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 7 & -1 & 3 \\ 4 & 8 & 0 & -6 \\ 6 & -9 & -2 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 9 & 6 & 0 \\ 2 & 3 & 8 & 4 \\ -5 & -2 & 10 & -3 \\ 7 & 4 & -2 & -5 \end{bmatrix}$ (iv) $\begin{bmatrix} -1 & 9 & 6 & 8 \\ -2 & 10 & 4 & 6 \\ 5 & 3 & 0 & 7 \\ 7 & -2 & 8 & 4 \end{bmatrix}$

[Madurai B.Sc. (Comp. Sc.) 92]

[**Ans.** (i) minimax = 3, maximin = 1, (ii) minimax = 0, maximin = -1, (iii) $2 < v < 4$, (iv) $4 \le v < 7$]

9.10. EQUIVALENCE OF RECTANGULAR GAME AND LINEAR PROGRAMMING

It has been shown that the player A chooses his optimum mixed stratgies in order to *maximize his minimum 'expected' gain, i.e.*

$$\max_{x_i} \left[\min \left\{ \sum_{i=1}^{m} v_{i1} x_i , \sum_{i=1}^{m} v_{i2} x_i , \dots , \sum_{i=1}^{m} v_{in} x_i \right\} \right] \qquad \text{...(9·16)}$$

subject to the constraints :

$$x_1 + x_2 + x_3 + \dots + x_m = 1 \qquad \text{...(9·17)}$$

$$x_i \ge 0, i = 1, 2, \dots , m . \qquad \text{...(9·18)}$$

Now, in order to express this problem in linear programming form, let

$$\min \left[\sum_{i=1}^{m} v_{i1} x_i , \sum_{i=1}^{m} v_{i2} x_i , \dots , \sum_{i=1}^{m} v_{in} x_i \right] = v \qquad \text{...(9·19)}$$

which immediately implies that

$$\sum_{i=1}^{m} v_{i1} x_i \ge v , \sum_{i=1}^{m} v_{i2} x_i \ge v , \dots , \sum_{i=1}^{m} v_{in} x_i \ge v . \qquad \text{...(9·20)}$$

Thus, the problem now becomes :Maximize $x_0 = v$ subject to the constraints :

$$\left.\begin{array}{lr} & v_{11}\,x_1 + v_{21}\,x_2 + v_{31}\,x_3 + \ldots + v_{m1}\,x_m \geq v \\ & v_{12}\,x_1 + v_{22}\,x_2 + v_{32}\,x_3 + \ldots + x_{m2}\,x_m \geq v \\ & \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ & v_{1n}\,x_1 + v_{2n}\,x_2 + v_{3n}\,x_3 + \ldots + v_{mn}\,x_m \geq v \\ & x_1 + x_2 + x_3 + \ldots + x_m = 1 \\ \text{and} & x_1\,,x_2\,,x_3\,,\ldots,x_m \geq 0 \end{array}\right\} \qquad \ldots(9{\cdot}21a)$$

Here v represents the value of the game. This linear programming formulation can be simplified by dividing all $(n+1)$ constraints by v ; the division is valid as long as $v > 0$* . In case, $v < 0$, the direction of the inequality constraints must be reversed, and if $v = 0$, division would be meaningless. The later point creates no special difficulty since a constant c can be added to all entries of the matrix ensuring that the value (v) of the game for the 'revised' matrix becomes greater than zero. After the optimal solution is obtained, the true value of the game is obtained by subtracting the same amount c.

In general, if the maximum value of the game is non-negative, the value of the game is greater than zero (provided the game does not have a saddle point). Thus, assuming $v > 0$, the constraints become :

$$\left.\begin{array}{r} v_{11}\dfrac{x_1}{v} + v_{21}\dfrac{x_2}{v} + \ldots + v_{m1}\dfrac{x_m}{v} \geq 1 \\ v_{12}\dfrac{x_1}{v} + v_{22}\dfrac{x_2}{v} + \ldots + v_{m2}\dfrac{x_m}{v} \geq 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n}\dfrac{x_1}{v} + v_{2n}\dfrac{x_2}{v} + ,\ldots + v_{mn}\dfrac{x_m}{v} \geq 1 \\ \dfrac{x_1}{v} + \dfrac{x_2}{v} + \ldots + \dfrac{x_m}{v} = \dfrac{1}{v} \end{array}\right\} \qquad \ldots(9{\cdot}21b)$$

Now, suppose $\dfrac{x_1}{v} = X_1\,,\dfrac{x_2}{v} = X_2\,,\ldots,\dfrac{x_m}{v} = X_m$, and $\dfrac{1}{v} = x_0$, then

$$\max v = \min\left(\frac{1}{v}\right) = \min\left\{\frac{x_1}{v} + \frac{x_2}{v} + \ldots + \frac{x_m}{v}\right\} \qquad \ldots(9{\cdot}22)$$

$= \min\,\{X_1 + X_2 + X_3 + \ldots + X_m\}$(which is justified by the last constraint),

Now, finally, the equivalent LP problem becomes :

Minimize $x_0 = X_1 + X_2 + \ldots + X_m$, subject to the constraints : ...(9·23)

$$\left.\begin{array}{r} v_{11}\,X_1 + v_{21}\,X_2 + \ldots + v_{m1}\,X_m \geq 1 \\ v_{12}\,X_1 + v_{22}\,X_2 + \ldots + v_{m2}\,X_m \geq 1 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ v_{1n}\,X_1 + v_{2n}\,X_2 + \ldots + v_{mn}\,X_m \geq 1 \\ X_1 \geq 0\,,X_2 \geq 0\,,\ldots,X_m \geq 0 \end{array}\right\} \qquad \ldots(9{\cdot}21c)$$

After an optimal solution is obtained by the simplex method, original optimal values can be obtained from the given transformation formulae.

On the other hand, player B chooses his mixed strategies in order to *minimize his maximum 'expected' loss, i.e.*

$$\min_{y_j}\left[\max\left\{\sum_{j=1}^{n} v_{1j}\,y_j\,,\sum_{j=1}^{n} v_{2j}\,y_j\,,\ldots,\sum_{j=1}^{n} v_{mj}\,y_j\right\}\right] \qquad \ldots(9{\cdot}24)$$

subject to the constraints :

$$y_1 + y_2 + \ldots + y_n = 1 \qquad \ldots(9{\cdot}25)$$

$$y_1 \geq 0\,,y_2 \geq 0\,,\ldots,y_n \geq 0. \qquad \ldots(9{\cdot}26)$$

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* For convenience, in order to convert a matrix game into a linear programming problem, first make all entries of the matrix positive by adding a positive constant c to all elements of the matrix game. Of course, c will be subtracted later on from the value of the game v.

Proceeding in the like manner, linear programming form of the B's problem becomes :

Maximize $\mathbf{y}_0 = Y_1 + Y_2 + \ldots + Y_n$, subject to the constraints : ...(9·27)

$$\left.\begin{array}{l} v_{11}\,Y_1 + v_{12}\,Y_2 + \ldots + v_{1n}\,Y_n \le 1 \\ v_{21}\,Y_1 + v_{22}\,Y_2 + \ldots + v_{2n}\,Y_n \le 1 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ v_{m1}\,Y_1 + v_{m2}\,Y_2 + \ldots + v_{mn}\,Y_n \le 1 \\ \quad Y_1 \ge 0\,, Y_2 \ge 0\,, \ldots, Y_n \ge 0 \end{array}\right\} \qquad \ldots(9{\cdot}28)$$

where $\mathbf{y}_0 = \dfrac{1}{v}\,,\; Y_1 = \dfrac{y_1}{v}\,, Y_2 = \dfrac{y_2}{v}\,, \ldots, Y_n = \dfrac{y_n}{v}\,.$

Further, it has been observed that the player B's problem is exactly the dual of the player A's problem. The optimal solution of one problem will automatically give the optimal solution to the other and that min $\mathbf{x}_0$ = max $\mathbf{y}_0$. The player B's problem can be solved by regular simplex method while player A's problem can be solved by the dual simplex method.

The choice of either method will depend on which problem has a smaller number of constraints. This in turn depends on the number of pure strategies for either player.

Q. 1. Show how a 'game' can be formulated as a linear programming problem. [IAS (Maths.) 99; Raj. Univ. (M. Phil) 90]

2. With the help of an appropriate example establish the relationship between 'Game theory' and 'Linear Programming'.

3. Establish the relation between a linear programming problem and a two-person zero-sum game. [Meerut (OR) 2003]

4. Discuss equivalence of matrix game and the problem of linear programming. [Kanpur M.Sc. (Math.) 97; Delhi (OR.) 95; Banasthali (M.Sc.) 93]

5. Explain the method of solving a zero-sum two person game as a linear programming problem. [Meerut 2005; Delhi 90]

6. Establish the equivalence of matrix game and the problem of linear programming. [Delhi B.Sc. (Math) 93]

9.11. MINIMAX THEOREM (FUNDAMENTAL THEOREM OF GAME THEORY)

Theorem 9·2. ***(Fundamental Theorem of Rectangular Games).*** *If mixed strategies are allowed, there always exists a value of the game, i.e.* $\overline{v} = \underline{v} = v$.

Alternative Statement. *If* $\Sigma x_i = \Sigma y_j = \overline{1}$, $x_i \ge 0$, $y_j \ge 0$, *then*

$$\max_{\mathbf{x}} \min_{\mathbf{y}\,|\,\mathbf{x}} \sum_i \sum_j v_{ij}\,(x_i\,y_j) = \min_{\mathbf{y}} \max_{\mathbf{x}\,|\,\mathbf{y}} \sum_i \sum_j v_{ij}\,(x_i\,y_j)\,,$$

where the symbol **y | x** *means "***y** *given* **x***". The left side relates that for some fixed (given)* **x**, *minimize the sum with respect to* **y**. *This results in a value showing it is a function of* **x**, *select* **x** *so that this value is maximum.*

Proof. The player A's problem (from sec. 9·10) is :

Min. $\mathbf{x}_0 = X_1 + X_2 + X_3 + \ldots + X_m$, subject to

$$\begin{array}{l} v_{11}X_1 + v_{21}X_2 + \ldots + v_{m1}X_m \ge 1 \\ v_{12}X_1 + v_{22}X_2 + \ldots + v_{m2}X_m \ge 1 \\ \quad\vdots \qquad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ v_{1n}X_1 + v_{2n}X_2 + \ldots + v_{mn}X_m \ge 1 \\ X_1 \ge 0, X_2 \ge 0, \ \ldots, X_m \ge 0\,. \end{array}$$

The dual problem corresponding to above linear programming problem (called the primal problem) is :

Max. $\mathbf{y}_0 = Y_1 + Y_2 + Y_3 + \ldots + Y_n$, subject to

$$\begin{array}{l} v_{11}Y_1 + v_{12}Y_2 + \ldots + v_{1n}Y_n \le 1 \\ v_{21}Y_1 + v_{22}Y_2 + \ldots + v_{2n}Y_n \le 1 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ v_{m1}Y_1 + v_{m2}Y_2 + \ldots + v_{mn}Y_n \le 1 \\ Y_1 \ge 0\,, Y_2 \ge 0\,, \ \ldots, Y_n \ge 0\,. \end{array}$$

It has been seen that this dual problem is similar to the problem obtained for the player *B* in *Sec. 9·10.*

But, the duality theorem states that :

If either the primal or the dual problem has a finite optimum solution, then the other problem has a finite optimum solution, and optimum numerical values of the objective function are equal, i.e.

$$\max \mathbf{y_0} = \min \mathbf{x_0} \quad \text{or} \quad \underline{v} = \overline{v} = v \text{ (value of the game)}$$

This completes the proof of the theorem.

Q. 1. State, explain and prove the 'minimax theorem' (fundamental theorem) for two-person zero-sum finite games.

[Kanpur M.Sc (Math.) 96; Delhi (OR) 93]

2. Let v be the value of a rectangular game with payoff matrix $B = (p_{ij})$. Show that $\min p_{ij} \le v \le \max (p_{ij})$ and $\max_i \min_j p_{ij} \le v \le \min_j \max_i p_{ij}$.

3. Let E (p, q) be expectation function in an m × n matrix rectangular game between player A and B, such that $p \in R^n$, $q \in R^m$. If E (p, q) be such that both $\max_p \min_q E(p, q)$ and $\min_q \max_p E(p, q)$ exist, then show that

$\min_q \max_p E(p, q) \ge \max_p \min_q E(p, q)$ (p and q are probability vectors)

[Raj. Univ. (M. Phil.) 91]

9·12 SOLUTION OF $m \times n$ GAMES BY LINEAR PROGRAMMING

Following example of (3 × 3) game will make the computational procedure clear.

Example 8. *Solve* (3 × 3) *game by the simplex method of linear programming whose payoff matrix is given below.*

Player *B*

		1	2	3
	1	3	−1	−3
Player *A*	2	−3	3	−1
	3	−4	−3	3

[JNTU (B. Tech.) 2004; Meerut (MCA) 2000]

Solution. First apply *minimax (maximin)* criterion to find the minimax ($\overline{v}$) and maximin ($\underline{v}$) value of the game. Thus, the following matrix is obtained (Table 9·8).

Table 9·8

B

		1	2	3	Row Minimum.
	1	$\boxed{3}$	−1	−3	−3
A	2	−3	$\boxed{3}$	−1	−3
	3	−4	−3	$\boxed{3}$	−4
Column Maximum		3	3	3	

Maximin Value ($\underline{v}$)

Minimax Value ($\overline{v}$)

Since, maximin value is − 3, it is possible that the value of the game (v) may be negative or zero because

$-3 < v < 3.$

Thus, a constant *c* is added to all elements of the matrix which is *at least* equal to the − ve of the maximin value, *i.e.* $c \ge 3$. Let $c = 5$. The matrix is shown in *Table* 9·9. Now, following the reasoning of *Sec.* **9·10**, the player *B*'s linear programming problem is :

Table 9.9

B

	1	8	4	2
A	2	2	8	4
	3	1	2	8

$$\text{Maximize } \mathbf{y}_0 = Y_1 + Y_2 + Y_3 \qquad \ldots(9{\cdot}29)$$

subject to the constraints :

$$8Y_1 + 4Y_2 + 2Y_3 \le 1 \,,\ 2Y_1 + 8Y_2 + 4Y_3 \le 1 \,,\ 1Y_1 + 2Y_2 + 8Y_3 \le 1 \,,\ Y_1 \ge 0 \,,\ Y_2 \ge 0,\ Y_3 \ge 0 \qquad \ldots(9{\cdot}30)$$

Introducing slack variables, the constraint equations become :

$$\left.\begin{array}{l} 8Y_1 + 4Y_2 + 2Y_3 + Y_4 \qquad\qquad = 1 \\ 2Y_1 + 8Y_2 + 4Y_3 \qquad + Y_5 \qquad = 1 \\ 1Y_1 + 2Y_2 + 8Y_3 \qquad\qquad + Y_6 = 1 \end{array}\right\} \qquad \ldots(9{\cdot}31)$$

$$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 \geq 0.$$

Now the following simplex table is formed.

Table 9·10. Simplex Table

		$c_j \to$	1	1	1	0	0	0	
B	C_B	Y_B	α_1 ↑	α_2	α_3	α_4 (β_1)	α_5 (β_2)	α_6 (β_3)	Min. Ratio (Y_B/α_k)
α_4	0	1	← $\boxed{8}$	4	2	1	0	0	1/8 ←
α_5	0	1	2	8	4	0	1	0	1/2
α_6	0	1	1	2	8	0	0	1	1/1
	$y_0 = C_BY_B = 0$		(– 1)* ↑	– 1	– 1	0 ↓	0	0	← $\Delta_j = C_B\alpha_j - c_j$
α_1	1	1/8	1	1/2	1/4	1/8	0	0	1/2
α_5	0	3/4	0	7	7/2	– 1/4	1	0	3/14
α_6	0	7/8	0	3/2	← $\boxed{31/4}$	1/8	0	1	7/62 ←
	$y_0 = 1/8$		0	– 1/2	(– 3/4)* ↑	1/8	0	0 ↓	← Δ_j
α_1	1	3/31	1	14/31	0	4/31	0	– 1/31	3/14
α_5	0	11/31	0	← $\boxed{196/31}$	0	– 6/31	1	14/31	11/196 ←
α_3	1	7/62	0	6/31	1	– 1/62	0	4/31	7/12
	$y_0 = 13/62$		0	(– 11/31)* ↑	0	7/62	0 ↓	3/31	← Δ_j
α_1	1	1/14	1	0	0	1/7	1/14	0	
α_2	1	11/196	0	1	0	– 3/98	31/196	– 1/14	
α_3	1	5/49	0	0	1	– 1/98	– 3/98	1/7	
	$y_0 = 45/196$		0	0	0	5/49	11/196	1/14	← all $\Delta_j \geq 0$

Thus, the solution for *B*'s original problem is obtained as :

$$y_1{}^* = \frac{Y_1}{y_0} = \frac{1/14}{45/196} = \frac{14}{45},\ y_2{}^* = \frac{Y_2}{y_0} = \frac{11/196}{45/196} = \frac{11}{45}.$$

$$y_3{}^* = \frac{Y_3}{y_0} = \frac{5/49}{45/196} = \frac{20}{45},\ v^* = \frac{1}{y_0} - c = \frac{196}{45} - 5 = -\frac{29}{45}.$$

The optimal strategies for the player *A* are obtained from the final table of the above problem. This is given by duality rules :

$$x_0 = y_0 = \frac{45}{196},\ X_1 = \Delta_4 = \frac{5}{49},\ X_2 = \Delta_5 = \frac{11}{196},\ X_3 = \Delta_6 = \frac{1}{14}.$$

Hence,
$$x_1{}^* = \frac{X_1}{x_0} = \frac{20}{45},\ x_2{}^* = \frac{X_2}{x_0} = \frac{11}{45},\ x_3{}^* = \frac{X_3}{x_0} = \frac{14}{45},\ v^* = \frac{29}{45}.$$

EXAMINATION PROBLEMS

1. Two companies *A* and *B* are competing for the same product. Their different strategies are given in the following payoff matrix :
Use linear programming to determine the best strategies for both the players.

[Madurai BSc (Math.) 93; Raj. (M. Phil.) 91]

$$\begin{array}{cc|ccc|} & & & A & \\ & & A_1 & A_2 & A_3 \\ \hline & B_1 & 2 & -2 & 3 \\ B & & & & \\ & B_2 & -3 & 5 & -1 \\ \hline \end{array}$$

[Hint. First, make the payoff's positive by adding a constant quantity $c = 4$ (say). The modiefied payoff matrix becomes

$$\begin{array}{cc} & A \\ B & \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix} \end{array}$$

Then, formulate the problem for player B by usual transformation as :

$$\text{Maximize } y_0 = Y_1 + Y_2., \text{ subject to} : 6Y_1 + Y_2 \le 1,\ 2Y_1 + 9Y_2 \le 1,\ 7Y_1 + 3Y_2 \le 1, \text{ and } Y_1 \ge 0,\ Y_2 \ge 0$$

Now apply simplex method to find the following solution for B :

$$v = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.\ y_1 = \frac{Y_1}{y_0} = \frac{7}{52} \times \frac{13}{3} = \frac{7}{12}.\ y_2 = \frac{Y_2}{y_0} = \frac{5}{52} \times \frac{13}{3} = \frac{5}{12}.$$

For player A, read the solution to the dual of above problem

$$v = \frac{1}{x_0} - 4 = \frac{1}{y_0} - 4 = \frac{13}{3} - 4 = \frac{1}{3}.\ x_1 = \frac{X_1}{x_0} = \frac{2}{13} \times \frac{13}{3} = \frac{2}{3}.\ x_2 = \frac{X_2}{x_0} = \frac{1}{13} \times \frac{13}{3} = \frac{1}{3}.\ x_3 = 0.]$$

[**Ans.** (2/3 , 1/3 , 0) ; (7/12 , 5/12) ; v = 1/3]

2. For the following payoff table, transform the zero-sum game into an equivalent linear programming problem and solve it by simplex method :

		Player Q		
		Q_1	Q_2	Q_3
Player P	P_1	9	1	4
	P_2	0	6	3
	P_3	5	2	8

[**Hint.** Payoff's are already non-negative. Formulation of L.P. problem for Q in usual notations is :

$$\text{Max. } y_0 = Y_1 + Y_2 + Y_3, \text{ subject to} :$$

$$9Y_1 + 1Y_2 + 4Y_3 \le 1,\ 0Y_1 + 6Y_2 + 3Y_3 \le 1,$$

$$5Y_1 + 2Y_2 + 8Y_3 \le 1, \text{ and } Y_1, Y_2, Y_3 \ge 0.$$

Its dual is the formulation for player P. Proceeding exactly as in solved example apply simplex method.

[**Ans.** (3/8 , 13/24 , 1/12) ; (7/24 , 5/9 , 11/72) ; v = 91/24]

3. Solve the following games by linear programming :

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

(ii)

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & 2 \\ 3 & 3 & -3 \end{bmatrix} \end{array}$$

(iii)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \end{array}$$

(iv)

$$\begin{array}{cc} & B \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & 4 \\ 2 & 3 \\ 3 & 2 \end{bmatrix} \end{array}$$

[**Ans.** *(i)* A (17/46 , 20/46 , 9/46); B (7/23 , 6/23 , 10/23) , v = 15/23] *(ii)* (6/11 , 3/11 , 2/11) ; . (5/22 , 8/22 , 9/22) ; v = 6/11] *(iii)* (2/3 , 1/3 , 0), (0, 1/2, 1/2), v = 1]

4. Solve the following 3 × 3 games by linear programming :

(i)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix} \end{array}$$

[Agra 98, 93, 92]

[**Ans.** A (6/13, 3/13 , 4/13), B (6/13, 4/13, 3/13), $v^* = -1/13$]

(ii)

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

[Meerut 93]

[**Ans.** (0, 0, 1), (4/5 , 1/5 , 0), v = 2]

5. A and B play a game in which each has three coins : a penny, a nickel and a dime. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, A wins *B's coins; If the sum is even,* B wins A's coin. Find the best strategies for each player and the value of game.

$$\left[\textbf{Ans. } A \begin{pmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ 1/2 & 1/2 & 0 \end{pmatrix}, B \begin{pmatrix} \text{Penny} & \text{Nickel} & \text{Dime} \\ 2/3 & 1/3 & 0 \end{pmatrix}, v = 0 \right]$$

6. A and B play a game as follows :
They simultaneously and independently write one of the three numbers 1, 2 and 3. If the sum of the numbers written is even, B pays to A this sum in Rupees. If it is odd, A pays the sum to B in Rupees. Form the payoff matrix of player A and solve the game to find out the value of the game and probabilities of mixed strategies of A and B.

[**Ans.** $\begin{pmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{pmatrix}$, A (1/4, 1/2, 1/4), B (1/4, 1/2, 1/4), v = 0]

7. Convert the following problems into linear programming problem :

(i)

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 3 & -2 \\ 2 & 4 & 0 \\ 4 & 5 & 1 \end{bmatrix} \end{array}$$

(ii)

$$\begin{array}{cc} B & \\ A & \begin{bmatrix} 8 & 20 & -3 & 1 \\ 6 & 25 & 4 & 2 \\ 0 & -8 & 12 & 9 \\ 16 & 9 & 21 & 0 \end{bmatrix} \end{array}$$

8. For the following payoff matrix, find the value of the game and the strategies of players A and B by using linear programming :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -1 & 4 \\ 6 & 7 & -2 \end{bmatrix} \end{array}$$

9. Transform the following matrix games into their corresponding primal and dual linear programming problems. Hence solve them.

(a) $\begin{pmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \end{pmatrix}$ **[Delhi BSc (Maths) 91]**

[**Ans.** (1/2, 1/2), (3/5, 2/5, 0), $v = 1$]

(b) $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ **[Delhi B.Sc. (Math.) 91]**

[**Ans.** (1/2 , 1/2 , 0), (0, 1/2 , 1/2), v = 0]

10. Use simplex method to solve the following games :

(a) $\begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$

[**Ans.** (2/3 , 1/2, 0), (0, 1/2 ,1/2), v = 5]

(b) $\begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

[**Ans.** (0, 0, 1), (4/5, 1/5, 0), v = 2]

11. Transform the following matrix game into its corresponding primal and dual linear programming problems :

$$\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix}$$

[Delhi M.Sc. (OR) 92]

Solve one of these linear programming problems to obtain the value and the optimal strategies for the two players.
[**Ans. Primal.** Min. $x_0 = X_1 + X_2$, subject to

$5X_1 + 4X_2 \geq 1$, $4X_1 + 3X_2 \geq 1$, $3X_1 + 6X_2 \geq 1$, $X_1 + 5X_2 \geq 1$, and $X_1 \geq 0$, $X_2 \geq 0$

Dual : Max. $y_0 = Y_1 + Y_2 + Y_3$, subject to $5Y_1 + 4Y_2 + 3Y_3 + Y_4 \leq 1$, $4Y_1 + 3Y_2 + 6Y_3 + 5Y_4 \leq 1$,
$Y_i \geq 0$, $i = 1, 2, 3, 4$, and $c = 3$.

12. In a two person gaime each player simultaneously shows either one or two fingers. If the number of fingers match, player A wins a rupee from player B, otherwise A pays a rupee to B. Shuw that the payoff matrix for this game is :

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ Solve this game by reducing it to an L.P.P.

[**Ans.** (1/2, 1/2), (1/2, 1/2), v = 0]

13. Two players independently select one of 'mouse', 'cat', 'tiger'. and 'elephant' and simultaneously reveal their choices. It is known that the cat chases the mouse (for score 1), the tiger chases the cat (for score 2), the elephant chases the tiger (for score 3) and the mouse chases the elephant (for a score 4). All other combinations yield a zero score. Formulate the payoff matrix and determine the optimal strategies of the two players.
[**Hint.** The payoff matrix is skew-symmetric :

$$\begin{array}{l|cccc|} & \text{Mouse} & \text{Cat} & \text{Tiger} & \text{Elephant} \\ \text{Mouse} & 0 & -1 & 0 & 4 \\ \text{Cat} & 1 & 0 & -2 & 0 \\ \text{Tiger} & 0 & 2 & 0 & -3 \\ \text{Elephant} & -4 & 0 & 3 & 0 \end{array}$$

14. Solve by using L.P. process, whose pay-off matrix is

$$A \begin{bmatrix} 3 & 2 & 4 & 0 \\ 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 8 \end{bmatrix}$$

[Meerut (M.A.) 97]

15. For the following pay-off matrix, find the value of the game and the strategies of players A and B by using linear programming :

$$\begin{array}{cc|ccc|} & & & \text{Player } B & \\ & & 1 & 2 & 3 \\ \text{Player } A & 1 & 3 & -1 & 4 \\ & 2 & 6 & 7 & -2 \end{array}$$

[Delhi (M.B.A.) 96]

[**Ans.** The solution to the problem, therefore, is : $S_A = (9/14, 5/14)$, $S_B = (0, 3/7, 4/7)$, value of game = 13/7.

9·13. TWO-BY-TWO (2 × 2) GAMES WITHOUT SADDLE POINT

There are several methods for determining the optimal strategies and the value of the game. But, in most of the situations, the matrix game can be reduced to a 2 × 2 game (to be discussed later in *Secs.* 9·14 & 9·15). It is therefore worth- while to determine the solution of 2 × 2 game in the following theorem.

Theorem 9·3. *Show that for any zero-sum two-person game where optimal strategies are not pure strategies (i.e. there is no saddle point) and for which the player A's payoff matrix is*

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & y_1 & y_2 \\ \hline A & x_1 & v_{11} & v_{12} \\ & x_2 & v_{21} & v_{22} \\ \hline \end{array}$$

and optimal strategies (x_1, x_2) *and* (y_1, y_2) *are determined by*

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}, \quad \frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}.$$

and the value (v) *of the game to the player A is given by*

$$v = \frac{v_{11}v_{22} - v_{12}v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})}.$$

[Meerut 2002; Rohilkhand 92]

Proof. Let a mixed strategy for player A be given by (x_1, x_2) where $x_1 + x_2 = 1$. Thus if player B moves his first strategy, the net expected gain of A will be $E_1(x) = v_{11}x_1 + v_{21}x_2$;
and if B moves his second strategy, the net expected gain of A will be $E_2(x) = v_{12}x_1 + v_{22}x_2$.

But, player A wants to maximize his minimum expected gain. So the value of the game (v) must be minimum of $E_1(x)$ and $E_2(x)$, *i.e.* $E_1(x) \geq v$, $E_2(x) \geq v$.

Thus for the player A, we have to find $x_1 \geq 0, x_2 \geq 0$, and v to satisfy the following three relationships (as obtained in ***Sec.9·10***):

$$v_{11}x_1 + v_{21}x_2 \geq v, \quad \text{...(9·33)}$$

$$v_{12}x_1 + v_{22}x_2 \geq v, \quad \text{...(9·34)}$$

$$x_1 + x_2 = 1. \quad \text{...(9·35)}$$

For optimum strategies, inequalities (9·33) and (9·34) become strict equations, *i.e.*

$$v_{11}x_1 + v_{21}x_2 = v, \quad \text{...(9·36)}$$

$$v_{12}x_1 + v_{22}x_2 = v. \quad \text{...(9·37)}$$

Subtracting equation (9·37) from the equation (9·36), we get

$$(v_{11} - v_{12})\, x_1 + (v_{21} - v_{22})\, x_2 = 0. \quad \text{...(9·38)}$$

which gives

$$\frac{x_1}{x_2} = \frac{v_{22} - v_{21}}{v_{11} - v_{12}}. \quad \text{...(9·39)}$$

Hence, we evaluate x_1 and x_2 separately by using the equation (9·35),

$$x_1 = \frac{v_{22} - v_{21}}{v_{11} + v_{22} - (v_{12} + v_{21})} \quad \text{...(9·40)}$$

$$x_2 = 1 - x_1 = \frac{v_{11} - v_{12}}{(v_{11} + v_{22}) - (v_{12} + v_{21})}. \quad \text{...(9·41)}$$

The value of the game can be obtained by substituting the values of x_1 and x_2 in either of the equations (9·36) and (9·37) to obtain

$$v = \frac{v_{11}(v_{22} - v_{21})}{v_{11} + v_{22} - (v_{12} + v_{21})} + \frac{v_{21}(v_{11} - v_{12})}{v_{11} + v_{22} - (v_{12} + v_{21})} \text{ or } v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}. \quad \text{...(9·42)}$$

In the same manner for the player B, find $y_1 \geq 0$, $y_2 \geq 0$, and v to satisfy the following three relations:

$$v_{11}y_1 + v_{12}y_2 \leq v, \quad \text{...(9·43)}$$

$$v_{21}y_1 + v_{22}y_2 \leq v, \quad \text{...(9·44)}$$

$$y_1 + y_2 = 1. \quad \text{...(9·45)}$$

Here it should be remembered that the player B wants to minimize his maximum loss.

Again for optimum strategies of player B, consider the inequalities (9·43) and (9·44) as strict equations and obtain

$$\frac{y_1}{y_2} = \frac{v_{22} - v_{12}}{v_{11} - v_{21}}. \quad \text{...(9·46)}$$

Using the equation (9·45)

$$y_1 = \frac{v_{22} - v_{12}}{v_{11} + v_{22} - (v_{21} + v_{12})} \quad \text{...(9·47)}$$

$$y_2 = 1 - y_1 = \frac{v_{11} - v_{21}}{v_{11} + v_{22} - (v_{21} + v_{12})}. \quad \text{...(9·48)}$$

Substituting values of y_1 and y_2 in either of the equation (9·43) or (9·44), to obtain the value

$$v = \frac{v_{11}v_{22} - v_{21}v_{12}}{v_{11} + v_{22} - (v_{12} + v_{21})}, \qquad \text{...(9·49)}$$

which is the same as desired by the *minimax theorem.*

If ratios x_1/x_2 and y_1/y_2 are both positive, these will give acceptable values of x_1, x_2, y_1 and y_2. A solution satisfying all constraints including non-negativity, may be obtained.

This proves the requied results.

Further for such games, in a payoff matrix *the largest and second largest elements must lie on one of the diagonals.* This implies that there are only 8 possible orderings (instead of 24) of entries $v_{11}, v_{12}, v_{21}, v_{22}$ without saddle point.

These possibilities are :

$$\begin{Bmatrix} v_{11} \ge v_{22} \ge v_{12} \ge v_{21} \\ v_{11} \ge v_{22} \ge v_{21} \ge v_{12} \end{Bmatrix}, \begin{Bmatrix} v_{22} \ge v_{11} \ge v_{21} \ge v_{12} \\ v_{22} \ge v_{11} \ge v_{12} \ge v_{21} \end{Bmatrix}, \begin{Bmatrix} v_{12} \ge v_{21} \ge v_{11} \ge v_{22} \\ v_{12} \ge v_{21} \ge v_{22} \ge v_{11} \end{Bmatrix}, \begin{Bmatrix} v_{21} \ge v_{12} \ge v_{11} \ge v_{22} \\ v_{21} \ge v_{12} \ge v_{22} \ge v_{11} \end{Bmatrix},$$

It can be easily verified that, with all above orderings, ratios x_1/x_2 and y_1/y_2 are non-negative.

Remark. If these formulae for x_1, x_2, y_1, y_2 and v are applied to a 2 × 2 games with saddle point, these may give an incorrect solution.

Q. 1. For the game $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$, where a, b, c, d are all non-negative ≥ 0, prove that the optimal strategies are :

$$A : \left(\frac{c+d}{a+b+c+d}, \frac{a+b}{a+b+c+d}\right), B : \left(\frac{b+d}{a+b+c+d}, \frac{a+c}{a+b+c+d}\right) \text{ and } v = \frac{ad-bc}{a+b+c+d}$$

[Meerut M.Sc (Math.) 96]

2. Given the 2 × 2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose player *A* adopts the strategy (x, y); while *B* adopts the strategy (u, v) where x, y, u, v are all ≥ 0, such that $x + y = u + v = 1$.

(i) Express *A*'s expected gain *z* in terms of x, y, u, v and a, b, c, d.
(ii) What is the effect on *z* of adding the same constant *k* to each element of the payoff matrix ?
(iii) What is the effect on *z* of multiplying each element of payoff matrix by the same constant *k* ?
(iv) How are the optimal strategies affected by these operations on payoff matrix.

3. If $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a non-strictly determined matrix game, then show that either

(i) $a < b, a < c, d < c, d < b$ or (ii) $a > b, a > c, d > c, d > b$. **[Kanpur 2000]**

4. Prove that 2 × 2 matrix game is strictly determined only if its principal diagonal elements are either strictly greater or strictly smaller than the other elements.

5. If all the elements of the payoff matrix of a game are non-negative and every column of this matrix has at least one positive element, then the value of the corresponding game is positive.

6. What do you mean by saddle point of a two-person zero-sum game ? In a 2 × 2 game if the largest and second largest elements lie along a diagonal, then prove that the game has no saddle point.

7. Let (a_{ij}) be the payoff matrix for a two-person zero-sum game. Examine the game for saddle point under the following orderings of its elements :

(a) $a_{21} \ge a_{22} \ge a_{11} \ge a_{12}$ (b) $a_{11} \le a_{12} \le a_{21} \le a_{22}$ (iii) $a_{12} \le a_{22} \le a_{11} \le a_{21}$ (iv) $a_{22} \le a_{11} \le a_{12} \le a_{21}$

8. For a two-person zero-sum game, the payoff matrix for player *A* is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with no saddle point. Obtain the optimal strategies (x_1, x_2) and (y_1, y_2) respectively.

Note. Students are advised to solve 2 × 2 games without saddle point by originally constructing the relationship for both the players instead of using the formulae (9·40), (9·41), (9·47), (9·48) and (9·49) directly.

9·13-1. Arithmetic Method for (2 × 2) Games

Arithmetic method provides an easy technique for obtaining the optimum strategies for each player in (2 × 2) games without saddle point. This method consists of the following steps :

Step 1. Find the difference of two numbers in column I, and put it under the column II, neglecting the negative sign if occurs.

Step 2. Find the difference of two numbers in column II, and put it under the column I, neglecting the negative sign if occurs.

Step 3. Repeat the above two steps for the two rows also.

The values thus obtained are called the ***oddments.*** These are the frequencies with which the players must use their courses of action in their optimum strategies.

The above steps can be explained by the following example :

Example 8. *Two players A and B without showing each other, put on a table a coin, with head or tail up. A wins Rs. 8 when both the coins show head and Re. 1 when both are tails. B wins Rs. 3 when the coins do not match. Given the choice of being matching player (A) or non-matching player (B), which one would you choose and what would be your strategy ?*

Solution. The payoff matrix for A is found to be

Player B

		H	T		
Player A	H	8	-3	4	$\frac{4}{11+4}=\frac{4}{15}$
	T	-3	1	11	$\frac{11}{11+4}=\frac{11}{15}$

Since no saddle point is found, the optimal strategies will be the mixed strategies.

Step 1. Taking the difference of two numbers in column I, we find $8-(-3)=11$, and put it under column II.

Step 2. Taking the difference of two numbers in column II, we find $(-3-1)=-4$, and put the number 4 (neglecting the $-$ve sign) under column I.

Step 3. Repeat the above two steps for the two rows also.

Thus for optimum gains, player A must use strategy H with probability $4/15$ and strategy T with probability $11/15$, while player B must use strategy H with probability $4/15$ and strategy T with probabiltiy $11/15$.

Step 4. To obtain the value of the game any of the following expressions may be used.

Using B's oddments :

$$B \text{ plays } H, \text{ value of the game, } v=\text{Rs.}\frac{4\times 8+11\times(-3)}{11+4}=\text{Rs.}\left(-\frac{1}{15}\right)$$

$$B \text{ plays } T, \text{ value of the game } v=\text{Rs.}\frac{4\times(-3)+11\times 1}{11+4}=\text{Rs.}\left(-\frac{1}{15}\right).$$

Using A's oddments :

$$A \text{ plays } H, \text{ value of the game, } v=\text{Rs.}\frac{4\times 8+11\times(-3)}{4+11}=\text{Rs.}\left(-\frac{1}{15}\right)$$

$$A \text{ plays } T, \text{ value of the game, } v=\text{Rs.}\frac{4\times(-3)+11\times 1}{4+11}=\text{Rs.}\left(-\frac{1}{15}\right).$$

The above values of v are equal only if the sum of the oddments vertically and horizontally are equal. Cases in which it is not so will be discussed later.

Thus the complete solution of the game is : (i) optimum strategy for A is $(4/15, 11/15)$. and for B is $(4/15, 11/15)$.

(ii) value of the game to A is $v=$ Rs. $(-1/15)$ and to B is $1/15$.

Thus, player A gains Rs. $(-1/15)$, *i.e.*, he loses Rs. $(1/15)$ which B, in turn, gets.

Note. Arithmetic method is easier than the algebraic method but it cannot be applied to larger games.

EXAMINATION PROBLEMS

Solve the following 2×2 games without saddle points :

1. $\begin{array}{cc} & B \\ A & \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$

2. $\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix} \end{array}$

3. $\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix} \end{array}$

4. $\begin{array}{cc} & B \\ A & \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \end{array}$

[**Ans.** 1. (1/5, 4/5), (3/5, 2/5) , $v=17/5$] 2.. (1/4, 3/4) for both player, $v=-3/4$] 3. (4/7, 3/7), (2/7, 5/7), $v=29/7$ 4. (1/3, 2/3), (3/5, 2/5), $v=0$]

5. $\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \end{array}$

6. $\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$

7. $\begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$

[**Ans.** (1/2, 1/2), (1/2, 1/2), $v=1/2$]

[**Ans.** A (1/2, 1/2), B (1/2, 1/2), $v=0$]

8. Two players A and B match coins. If the coin match, then A wins one unit of value, if the coins do not match, then B wins one unit of value. Determine optimum strategies for the players and the value of the game.

[**Hint.** Formulation of the game is : $\begin{array}{ccc} & & B \\ & & \begin{array}{cc} H & T \end{array} \\ A & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \end{array}$ [**Ans.** (1/2, 1/2), (1/2, 1/2), $v=0$]

9. *A* and *B* each take out one or two matches and guess how many matches opponent has taken. If one of the players guess correctly then the looser has to pay him as many rupees as the sum of the numbers held by both players. Otherwise, the payout is zero. Write down the payoff matrix and obtain the optimal strategies of both players.

[Hint. Formulation of the game is :
$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} 1 & 2 \end{array} \\ A & \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \end{array}$$
[Ans. (2/3, 1/3), (2/3, 1/3), $v = 4/3$]

10. In a game of matching coins with two players, suppose *A* wins one unit of value when there are two heads, wins nothing when there are two tails, and loses 1/2 unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player, and the value of the game to *A*.

[Hint. Formulation of the game is :
$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} H & T \end{array} \\ A & \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \end{array}$$
[Ans. (1/4, 3/4), (1/4, 3/4), $v = -1/8$]

11. Consider a modified form of 'matching biased wins' game problem. The matching player is paid eight rupees if the two wins turn both heads and one rupee if the two wins turn both tails. The non-matching player is paid three rupees when the two wins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy ? **[IAS (Maths. 97]**

12. Solve the following game and determine the value of the game :

		Player Y	
		Strategy 1	Strategy 2
Player X	Strategy 1	4	1
	Strategy 2	2	3

[Allahabad (M.B.A.) 98]

[Ans. The optimum strategies for the two players are :
S_X = (1/4, 3/4) and S_Y = (1/2, 1/2) and the value of game = 10/4.]

9·14 PRINCIPLE OF DOMINANCE TO REDUCE THE SIZE OF THE GAME

For easiness of solutions, it is always convenient to deal with smaller payoff matrices. Fortunately, the size of the payoff matrix can be considerably reduced by using the so called *principle of dominance.* Before stating this principle, let us define a few important terms.

Inferior and Superior Strategies. *Consider two n-tuples* $\mathbf{a} = (a_1, a_2, \dots, a_n)$ *and* $\mathbf{b} = (b_1, b_2, \dots, b_n)$. *If* $a_i \geq b_i$ *for all* $i = 1, 2, \dots, n$, *then for player A the strategy corresponding to* **b** *is said to be* ***inferior*** *to the strategy corresponding to* **a**; *and equivalently, the strategy corresponding to* **a** *is said to* ***be superior*** *to the strategy corresponding to* **b.**

For player B, the above situation will be reversed, because player A's gain-matrix is player B's loss-matrix.

Dominance. *An n-tuple* $a = (a_1, a_2, \dots, a_n)$ *is said to* ***dominate*** *the n-tuple* $\mathbf{b} = (b_1, b_2, \dots, b_n)$ *if* $a_i \geq b_i$ *for all* $i = 1, 2, \dots, n$. *The superior strategies are said to dominate the inferior ones.*

Thus a player would not like to use inferior strategies which are dominated by other's. Now we are able to state the principle of dominance as follows :

Principle of Dominance. *If one pure strategy of a player is better or superior than another one (irrespective of the strategy emplayed by his opponent), then the inferior strategy may be simply ignored by assigning a zero probability while searching for optimal strategies.*

Theorem 9.4 (Dominance Property). *Let* $\mathrm{A} = [v_{ij}]$ *be the payoff matrix of an* $m \times n$ *rectangular game. If the ith row of* A *is dominated by the rth row of* A, *then the deletion of ith row of* A *does not change the set of optimal strategies for the row player (player* A*).*

Further, if the jth column of A *dominates the kth column of* A, *then the deletion of jth column of* A *does not change the set of optimal strategies for the column player (player B).*

Proof. Given that

$$v_{ij} \leq v_{rj}, \text{ for all } j = 1, 2, \dots, n \text{ and } v_{ij} \neq v_{rj} \text{ for at least one } j \quad \dots(1)$$

Let $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_n^*)$ be an optimal strategy for the column player B. It follows from (1) that

$$\sum_{j=1}^{n} v_{ij} y_j^* < \sum_{j=1}^{n} v_{rj}\, y_j^* \quad \text{or} \quad E(\mathbf{e}_i, \mathbf{y}^*) < E(\mathbf{e}_r, \mathbf{y}^*)$$

$$\therefore \qquad v \ge E(\mathbf{e}_r, \mathbf{y}^*) > E(\mathbf{e}_i, \mathbf{y}^*) \qquad ...(2)$$

Now let $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)$ be an optimal strategy for the row player. If possible, let us suppose $x_i^* > 0$, then from (2) $\quad x_i^*\, v > x_i^*\, E(\mathbf{e}_i, \mathbf{y}^*)$

Also, we have $$v = E(\mathbf{x}^*, \mathbf{y}^*) = \sum_{i=1}^{m} x_i^*\, E(\mathbf{e}_i, \mathbf{y}^*)$$

$$= x_r^*\, E(\mathbf{e}_r, \mathbf{y}^*) + \sum_{i \neq r}^{m} x_i^*\, E(\mathbf{e}_i, \mathbf{y}^*)$$

$$< x_r^* v + \sum_{i \neq r}^{m} x_i^* v = v \sum_{i=1}^{m} x_i^* = v \qquad \left(\therefore \sum_{i=1}^{m} x_i^* = 1\right)$$

which is a contradiction, and hence $x_i^* = 0$.

Second part can also be proved similarly.

9.14-1 Generalized Dominance Property

The dominance property is not only based on the superiority of pure strategies only, but on the superiority of some *convex linear combination* of two or more pure strategies also. A given strategy can also be said to be dominated if it is inferior to some convex linear combination of two or more strategies. This concept generalizes the above dominance principle in the following theorem.

Theorem. 9·5 (Generlized Dominance). *Let* $\mathbf{A} = [v_{ij}]$ *be the pay-off matrix of an* $m \times n$ *rectangular game. If the ith row of* **A** *is strictly dominated by a convex combination of the other rows of* **A**, *then the deletion of the ith row of* **A** *does not effect the set of optimal strategies for the row player (the player A).*

Further, if the jth column of **A** *strictly dominates a convex combination of the other columns, then the deletion of the jth column of* **A** *does not effect the optimal strategies for the column player (the player B).*

Proof. Let $\mathbf{A} = [v_{ij}]$ be the payoff matrix considering the first part, we are given that there exist scalars (probabilities) $x_1, x_2, \dots, x_m$ $(0 \le x_i \le 1, x_r = 0, \Sigma x_i = 1)$ such that

$$\sum_{i=1, i \neq r}^{m} x_i\, v_{ij} \ge v_{rj}, \text{ for } j = 1, 2, \dots, n$$

or $$\sum_{i=1}^{m} x_i\, v_{ij} \ge v_{rj}, \text{ for } j = 1, 2, \dots n \qquad (\because x_r = 0) \qquad ....(1)$$

where strict inequality holds for at least one j.

Let $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_n^*)$ be an optimal strategy for player B. Then it follows from (1) that

$$\sum_{j=1}^{n} v_{rj}\, y_j^* < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\, x_i\, y_j^*,$$

or $$E(\mathbf{e}_r, \mathbf{y}^*) < \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\, x_i\, y_j^* \le v \qquad ...(2)$$

Let $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_m^*)$ be an optimal strategy for player A.

If possible, let us suppose that $x_r^* \neq 0$ .From (2), we know that $E(\mathbf{e}_r, \mathbf{y}^*) < v$.

Then since $x_r^* \neq 0$, we must have $x_r^* E(\mathbf{e}_r, \mathbf{y}^*) < x_i^*\, v$.

Thus $$E(\mathbf{x}^*, \mathbf{y}^*) = \sum_{j=1}^{n} \sum_{i=1}^{m} v_{ij}\; x_i^*\, y_j^* = x_r^*\, E(\mathbf{e}_r, \mathbf{y}^*) + \sum_{i \neq r} x_i^*\, E(\mathbf{e}_i, \mathbf{y}^*)$$

This implies $$v < x_r^*\, v + v \sum_{i \neq r} x_i^* = v \sum_{i=1}^{m} x_i^* = v \text{ which is a contradiction.}$$

Hence we must have $x_r{}^* = 0$. This completes the proof for the first part.

Similarly, we can prove the second part.

Remarks:

1. If $v_{rj} = \sum_{1}^{m} x_i v_{ij}$ for all $j = 1, 2, ..., n$; the result follows trivially, for then any probability assigned to the rth row can be easily distributed over the other rows, and rth row itself is ignored.
2. It should also be noted here that the dominating column is deleted whereas the row dominated by a convex combination of other rows is deleted.

9·14–2. Summary of Dominance Rules

The *"dominance property"* can be summerized in the following rules :

Rule 1. If each element in one row, say rth of the payoff matrix $[v_{ij}]$, is less than or equal to the corresponding element in the other row, say sth, then the player A will never choose the rth strategy. In other words, if for all $j = 1, 2, ..., n$, and $v_{rj} \leq v_{sj}$, then the probability x_r of choosing rth strategy will be zero. The value of the game and the non-zero choice of probabilities remain unaltered even if rth row is deleted from the payoff matrix. Such rth row is said to be dominated by the sth row.

Rule 2. Following the similar arguments, if each element in one column, say C_r, is greater than or equal to the corresponding element in the other column, say C_s, then the player B will never use the stategy corresponding to column C_r. In this case, the column C_s dominates the column C_r.

Rule 3. Dominance need not be based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. In general, if some convex linear combination of some rows dominates the ith row, then the ith row will be deleted. If the ith row dominates the convex linear combination of some other rows, then one of the rows involving in the combination may be deleted. Similar arguments follow for columns also.

Rule 4. If (x_1, x_2) be the optimal strategy for the player A for the reduced game and (w_1, x_2) be the optimal strategy for the original game, then w_1 is the ith place extension of x_1.

Rule 5. If (y_1, y_2) be the optimal strategy for the player B for the reduced game and (y_1, w_2) be the optimal strategy for the original game, then w_2 is the jth place extension of y_2.

Rule 6. If the dominance holds strictly, then values of optimal strategies do coincide, and when the dominance does not hold strictly, then optimal strategies may not coincide.

Note. Using dominance properties, try to reduce the size of payoff matrix.

9·14–3. Demonstration of Dominance Properties by Examples

1. ***To illustrate first and second properties,*** consider the example of (3 × 3) game [*Table* 9·11].

It is clear that this game has no saddle point. However, consider Ist and IIIrd columns from player B's point of view. It is seen that each payoff (element) in IIIrd column is greater than the corrresponding element in Ist column regardless of the player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss as compared to that of selecting the Ist strategy. The column III is *inferior* to I as never to be used. Hence, deleting the IIIrd column which is dominated by I, the reduced-size payoff matrix (*Table 9·12*) is obtained.

Again, if the reduced matrix (*Table 9·12*) is looked from player A's point of view, it is seen that the player A will never use the II strategy which is dominated by III. Hence, the size of matrix can be reduced further by deleting the II row (*Table 9·12*). This reduced matrix can be further reduced by deleting II row as shown in *Table 9·13*. The solution of the reduced 2 × 2 matrix game without saddle point can be easily obtained by solving the following simultaneous equations in usual notations :

$$-4x_1 + 2x_3 = v\ ,\ 6x_1 - 3x_3 = v\ ,\ x_3 + x_1 = 1 \text{ (For } A\text{)}$$

and
$$-4y_1 + 6y_2 = v\ ,\ 2y_1 - 3y_2 = v\ ,\ y_1 + y_2 = 1 \text{ (For } B\text{)}$$

Table 9·11

		B		
		I	II	III
	I	−4	6	3
A	II	−3	−3	4
	III	2	−3	4

Table 9·12

		B	
		I	II
	I	−4	6
A	II	−3	−3
	III	2	−3

Table 9·13

		B	
		I	II
A	I	−4	6
	III	2	−3

It is advisable to verify the solution :

(i) The player A chooses mixed strategy $(x_1, x_2, x_3) = (1/3, 0, 2/3)$

(ii) The player B chooses mixed strategy $(y_1, y_2, y_3) = (3/5, 2/5, 0)$

(iii) The value of the game is zero, *i.e.*, the game is fair.

Q. 1. Explain the term 'saddle point' and 'dominance' used in game theory. **[Raj. Univ. (M. Phil) 89]**

2. Write short note on 'concept of dominance'. **[Kanpur M.Sc (Math.) 93]**

3. Explain the concept of generalized dominancy in the context of game theory. **[Kanpur M.Sc. (Math.) 97; Delhi M.Sc. (Stat.) 95]**

4. Briefly explain the general rules for dominance. **[JNTU (B. Tech.) 2004, 03]**

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2. ***To illustrate the third property***, consider the following game matrix [*Table* 9·*14*]

Table 9·14

		B		
		1	2	3
A	1	5	0	2
	2	−1	8	6
	3	1	2	3

None of the *pure* strategies of the player A is inferior to any of his other pure strategies. However, the average of the player A's first and second pure strategies gives

$$\left\{\frac{5-1}{2}, \frac{0+8}{2}, \frac{2+6}{2}\right\} \text{ or } (2, 4, 4)$$

Obviously, this is superior to the player A's third pure strategy. So the third strategy may be deleted from the matrix. The reduced matrix is shown in *Table* 9·15.

Table 9·15

	1	2	3
1	5	0	2
2	−1	8	6

Q. 1. Explain the following terms :

(i) Two-person zero-sum game, (ii) Principle of dominance, (iii) Pure strategy in game theory. **[Agra 94]**

2. How is the concept of dominance used in simplifying the solution of a rectangular game ? **[Meerut (OR) 2003]**

3. Explain the principle and rules of dominance to reduce the size of payoff matrix. **[VTU (BE Mech.) 2002]**

4. State the general rules of dominance for two person zero-sum games.

5. Let R_1, R_2 be the subsets of the rows of an $m \times n$ payoff matrix A. Likewise, let C_1, C_2 be the subsets of the columns of A. Show that if a convex combination of the rows (columns) in R_1 (C_1) dominates a convex combination of the rows (columns) in R_2 (C_2). Then there exists a row (column) in R_2 (C_1) which, if deleted, does not change the set of optimal strategies for player A (player B).

6. Show that the existence of a saddle point in 2 × 2 game implies the existence of a dominating pure strategy for at least one of the players and conversely.

Solved Examples

Example 9. *Solve the game whose payoff matrix to the player A is given in the table :*

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7
	III	5	2	6

[Rohit 91]

Solution. Since the row III is inferior to the row II, row III can be deleted from the payoff matrix. Thus the reduced matrix (*Table* 9·*16*) is obtained.

Table 9·16

		B		
		I	II	III
A	I	1	7	2
	II	6	2	7

Again, column III is dominated by column I, therefore column III can also be deleted from the above matrix. The reduced matrix is given in (*Table* 9·*17*).

Table 9·17

		B	
		(y_1) I	(y_2) II
A	(x_1) I	1	7
	(x_2) II	6	2

This 2 × 2 game without saddle point can be solved either by putting $v_{11} = 1, v_{12} = 7, v_{21} = 6, v_{22} = 2$ in the formulae of *Sec.* **9·13**, or by solving the simultaneous equations :

$1x_1 + 6x_2 = v$, $7x_1 + 2x_2 = v$, $x_1 + x_2 = 1$ ***(For player A)***

$1y_1 + 7y_2 = v$, $6y_1 + 2y_2 = v$, $y_1 + y_2 = 1$ ***(For player B)***

Thus the following solution is obtained :

(i) The player A chooses optimal strategy $(x_1, x_2, x_3) = (2/5, 3/5, 0)$.

(ii) The player B chooses optimal strategy $(y_1, y_2, y_3) = (1/2, 1/2, 0)$.

(iii) The value of the game to the player A is $v = 4$.

Example 10. *Use the relation of dominance to solve the rectangular game whose payoff matrix to A is given in Table 9·18.* [Raj. (M.Phil) 92; Delhi B.Sc. (Math) 93; I.C.W.A. (June) 92]

Table 9·18

B

		I	II	III	IV	V	VI
	I	0	0	0	0	0	0
	II	4	2	0	2	1	1
	III	4	3	1	3	2	2
A	IV	4	3	7	−5	1	2
	V	4	3	4	−1	2	2
	VI	4	3	3	−2	2	2

Solution. In the payoff matrix from player A's point of view, rows I and II are dominated by the row III. Hence the player A will never use strategies I and II in comparison to the strategy III. Thus, deleting I and II rows we obtain the reduced matrix (*Table 9·19*).

Table 9·19

	I	II	III	IV	V	VI
III	4	3	1	3	2	2
IV	4	3	7	−5	1	2
V	4	3	4	−1	2	2
VI	4	3	3	−2	2	2

Again, from the player B's point of view, columns I, II and VI are dominated by the column V. Therefore, the player B will never use strategies I, II and VI in comparison to the strategy V. Now, delete columns I, II and VI from the matrix to obtain the new matrix (*Table 9·20*).

Again the row VI is dominated by the row V from the player A's point of view. Hence, deleting VIth row, obtain the next reduced matrix. [*Table 9·21*]

Table 9·20

B

		III	IV	V
	III	1	3	2
A	IV	7	−5	1
	V	4	−1	2
	VI	3	−2	2

Table 9·21

	III	IV	V
III	1	3	2
IV	7	−5	1
V	4	−1	2

None of the pure strategies of the player B is inferior to any of his other strategies. However, the average of player B's III and IV pure strategies gives,

$$\left\{\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right\} \quad \text{or } (2, 1, 3/2)$$

which is obviously superior to the player B's Vth pure strategy, because Vth strategy will result much more losses to B. Thus deleting the Vth strategy from the matrix, the revised matrix (*Table 9·22*). is obtained :

Table 9·22

B

		III	IV
	III	1	3
A	IV	7	−5
	V	4	−1

Also, the average of the player A's III and IV pure strategies give

$$\left\{\frac{1+7}{2}, \frac{3-5}{2}\right\} \quad \text{or } (4, -1).$$

This is obviously the same as the player A's Vth strategy.

In this case, the Vth strategy may be deleted from the matrix. Finally, (2 × 2) reduced matrix (*Table 9·23*) is obtained.

Table 9·23

B

		(y_3) III	(y_4) IV
A	(x_3) III	1	3
	(x_4) IV	7	−5

Now, for (2 × 2) game, having no saddle point, solve the following simultaneous equations :

$1.x_3 + 7x_4 = v$, $3x_3 - 5x_4 = v$, $x_3 + x_4 = 1$ (*For A*)

$1.y_3 + 3y_4 = v$, $7y_3 - 5y_4 = v$, $y_3 + y_4 = 1$ (*For B*)

The solution is :

(i) The player A chooses the optimal strategy (0, 0, 6/7, 1/7, 0, 0).

(ii) The player B chooses the optimal strategy (0, 0, 4/7, 3/7, 0, 0).

(iii) The value of the game to player A is 13/7 .

Example 11. *Two competitors A and B are competing for the same product. Their different strategies are given in the following payoff matrix:*

Table 9·24 (a)

		Company B			
		I	II	III	IV
Company A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

use dominance principle to find the optimal strategies. **[JNTU (B. Tech.) 2003 Type; Meerut 02; Delhi (Stat.) 95, B.Sc. (Math.) 90; Rohilkhand 94, 93; Banasthali 93; Kanpur 93]**

Solution. First, we can find that this game does not have a saddle point. Now try to reduce the size of the given payoff matrix by using the principle of dominance.

From the player A's point of view, Ist row is dominated by the IIIrd row. So delete Ist row from the matrix [*Table 9·24 (a)*].

Again, from the player B's point of view, 1st column is dominated by the IIIrd column. Hence, Ist column may also be deleted from the matrix. Thus, the reduced payoff matrix [*Table 9·24 (b)*] is obtained.

Table 9·24 (b)

		B		
		II	III	IV
A	II	4	2	4
	III	2	4	0
	IV	4	0	8

In order to check the further reduction of this reduced matrix, the average of the player B's III and IV pure strategies give

$$\left\{\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (3, 2, 4)$$

which is obviously superior to the player B's II pure strategy. Under this condition, the player B will not use II strategy. Hence, II column may be deleted from the matrix. Thus, new matrix (*Table 9·25*), is obtained.

Table 9·25

		B	
		III	IV
A	II	2	4
	III	4	0
	IV	0	8

Again, in the new matrix, the average of the player A's III and IV pure strategies give

$$\left\{\frac{4+0}{2}, \frac{0+8}{2}\right\} \text{ or } (2, 4)$$

which is obviously the same as the player A's II strategy. Therefore, the player A will gain the same amount even if the II strategy is never used by him. Hence deleting the player A's II strategy from the matrix to obtain the reduced (2 × 2) matrix (*Table 9·26*).

Table 9·26

			B	
			(y_3) III	(y_4) IV
A	x_3	III	4	0
	x_4	IV	0	8

Since this (2 × 2) payoff matrix has no saddle point, solve the simultaneous equations :

$4.x_3 + 0x_4 = v$, $0x_3 + 8x_3 = v$, $x_3 + x_4 = 1$ (*For player A*)

$4.y_3 + 0y_4 = v$, $0y_3 + 8y_4 = v$, $y_3 + y_4 = 1$ (*For player B*)

to get the solution :

(i) Optimal strategy for the player $A = (x_1 , x_2 , x_3 , x_4) - (0, 0, 2/3 , 1/3)$.

(ii) Optimal strategy for the player $B = (y_1 , y_2 , y_3 , y_4) = (0, 0, 2/3, 1/3)$.

(iii) The value of the game to the player A is $v = 8/3$.

Example 12. *Solve the following game using dominance principle :*

$$\begin{array}{cc|ccccc|} & & & & \text{Player } B & & \\ & & \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \\ & \text{I} & 3 & 5 & 4 & 9 & 6 \\ \text{Player } A & \text{II} & 5 & 6 & 3 & 7 & 8 \\ & \text{III} & 8 & 7 & 9 & 8 & 7 \\ & \text{IV} & 4 & 2 & 8 & 5 & 3 \end{array}$$

Solution. In the given payoff matrix, IVth column dominates the Ist column and also Vth column dominates the IInd column. So Ist and IInd columns can be deleted without affecting the optimal strategies of B. Thus we get the reduced payoff matrix (a).

Again, we observe that IIIrd row of the reduced matrix dominates all the other rows. Thus the payoff matrix (b) is obtained.

Again, the IInd column of (b) is dominated by both the Ist and IIIrd columns. Thus the reduced payoff matrix (c) is obtained.

$$\begin{array}{cc|ccc|} & & & B & \\ & & \text{I} & \text{II} & \text{III} \\ \cline{3-5} & \text{I} & 3 & 5 & 4 \\ A & \text{II} & 5 & 6 & 3 \\ & \text{III} & 8 & 7 & 9 \\ & \text{IV} & 4 & 2 & 8 \\ \cline{3-5} \end{array} \qquad \begin{array}{c|c|c|c|} & \text{I} & \text{II} & \text{III} \\ \cline{2-4} \text{III} & 8 & 7 & 9 \\ \cline{2-4} \end{array} \qquad \begin{array}{c|c|} & \text{II} \\ \cline{2-2} \text{III} & 7 \\ \cline{2-2} \end{array}$$

(a) (b) (c)

Thus the solution to the game is :

(i) best strategy for player A is III; *(ii)* best strategy for player B is II; and

(iii) value of the game for player A is 7, and for player B is – 7.

Note. If we apply principle of dominance to the payoff matrix having a saddle point, then we get a single element reduced matrix only. So the students are advised to use the principle of dominance for solving the games without saddle point until unless otherwise stated.

EXAMINATION PROBLEMS

Explain the principle of dominance and hence solve the following games :

1.

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Player } A \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 6 & 8 & 6 \\ 4 & 12 & 2 \end{bmatrix} \end{array}$$

[**Ans.** (I, I) ; (II, III); $v = 6$]

2.

$$\begin{array}{cc} & B \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ A \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -5 & 3 & 1 & 20 \\ 5 & 5 & 4 & 6 \\ -4 & 2 & 0 & -5 \end{bmatrix} \end{array}$$

[Raj. Univ. (M. Phil) 90]

[**Ans.** (2, III), $v = 4$]

3.

$$\begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix}$$

[Delhi. (OR.) 95]

4.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 3 & 1/2 \\ 3/2 & 2 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \end{array}$$

[**Ans.** (1/4, 0, 3/4) for both players, $v = 7/8$]

5.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 8 & 4 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

6. Use dominance principle to reduce the following games to 2 × 2 games and hence solve them.

(i) $\begin{pmatrix} 2 & 0 & 3 \\ 3 & -1 & 1 \\ 5 & 2 & -1 \end{pmatrix}$ *(ii)* $\begin{pmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{pmatrix}$ *(iii)* $\begin{pmatrix} 8 & 5 & 8 \\ 8 & 6 & 5 \\ 7 & 4 & 5 \\ 6 & 5 & 6 \end{pmatrix}$ *(iv)* $\begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$

[Delhi B.Sc. (Math.) 91]

[**Ans.** *(i)* (1/2, 0, 1/2), (2/3, 1/3, 0), $v = 0$, *(ii)* (0, 7/12, 5/12), (0, 1/3, 2/3), $v = -1/3$.
(iii) (1/4, 1/4, 0, 0), (0, 3/4, 1/4), $v = 23/4$, *(iv)* (0, 0, 1), (2/5 , 3/5, 0), $v = 2$].

7.

$$\begin{array}{cc} & \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 4 & 4 & 2 & -4 & -6 \\ 8 & 6 & 8 & -4 & 0 \\ 10 & 2 & 4 & 10 & 12 \end{bmatrix} \end{array}$$

[**Ans.** A (0, 4/9, 5/9), B (0, 7/9, 0, 2/9, 0) = $v = 34/9$]

8. Following is the payoff matrix for player A :

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 4 & 3 & 8 & 4 \\ 5 & 6 & 3 & 7 & 8 \\ 6 & 7 & 9 & 8 & 7 \\ 4 & 2 & 8 & 4 & 3 \end{bmatrix} \end{array}$$

Using dominance properties, obtain the optimum strategies for both the players and determine the value of the game.
[**Ans.** (III, I), $v = 6$ for A]
Note. Also, without using dominance property saddle point (6) exist in this problem.

9. A and B play a game in which each has three coins, a 5 p., a 10 p., and a 20 p. Each selects a coin without the knowledge of the others choice. If the sum of the coins is an odd amount, A wins B's coin; if the sum is even B wins A's coin. Find the best strategy for each player and the value of game. **[Kanpur 2000, 97]**

[**Hint.** Formulation of the game is as follows :

$$\begin{array}{cc} & B \\ A & \begin{array}{c|ccc|} & 5P & 10P & 20P \\ 5P & -5 & 10 & 20 \\ 10P & 5 & -10 & -10 \\ 20P & 5 & -20 & -20 \end{array} \end{array}$$

[**Hint.** Reduce this game to 2 × 2 using dominance. III row is dominated by II, then III column in dominated by II].
[**Ans.** (1/2, 1/2, 0), (2/3, 1/3, 0), $v = 0$].

10. In a small town there are two discount stores *ABC* and *XYZ*. They are the only stores handle sundry goods. The total number of customers is equally divided between the two, because the price and quality of goods sold are equal. Both stores have good reputations in the community, and they render equally good customer services. Assume that a gain of customers by *ABC* is a loss to XYZ, and *vice-versa*.

		Strategies of *XYZ*		
		Press	Radio	Television
Strategies of *ABC*	Press	30	40	-80
	Radio	0	15	-20
	Television	90	20	50

Both stores plan to run annual pre-diwali sales during the first week of October. Sales are advertised through the local newspaper, radio and television media. With the aid of an advertising firm, *ABC* store constructed the same matrix given below (Figures in the matrix represent a gain or loss of customer). Find the optimal strategies for both stores and the value of the game.

[**Hint.** Use dominance to reduce the size to 2 × 2 game.

Ans. $\begin{pmatrix} \text{Press} & \text{Radio} & \text{T.V.} \\ 1/5 & 0 & 4/5 \end{pmatrix}, \begin{pmatrix} \text{Press} & \text{Radio} & \text{T.V.} \\ 0 & 13/15 & 2/15 \end{pmatrix}, v = 24$]

11. **(Media Problem).** Consider the following payoff matrix for two firms. What is the best mixed strategy for both the firms and also find out the value of the game ?

		Firm II		
		No advertising	Medium advertising	Large advertising
Firm I	No advertising	60	50	40
	Medium adversiging	70	70	50
	Large advertising	80	60	75

[Delhi (M.Com.) 90 Type]

[**Hint.** Use dominance to reduce it to 2 × 2 form, then solve].
[**Ans.** (0, 3/7 , 4/7), (0, 5/7, 2/7), $v = 150/7$]

12. Explain the terms 'Saddle point' and 'Dominance' in connection with the theory of games. Show that if dominance occurs in the payoff matrix of 2 × 2 game, then there is a saddle point. Is the converse true ? Solve the game whose payoff matrix is as given below.

$$\begin{bmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 2 & 6 \end{bmatrix}$$

[**Hint.** Use dominance to reduce it to 2 × 2 form, then solve]. **[Raj. (M. Phil.) 93. 92]**

13. Given the payoff matrix for player *A*, obtain the optimum strategies for both the players and determine the value of the game. **[Jammu (M.B.A.) Jan. 99]**

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 6 & -3 & 7 \\ -3 & 0 & 4 \end{bmatrix} \end{array}$$

[**Ans.** Optimum strategies for players *A* and *B* will be as follows :

S_A = [1/4, 3/4] and S_B = [1/4, 3/4, 0]

Expected value of game = $\frac{3}{4}$.]

14. *A* is paid Rs. 8·00 if two coins turn both heads and Re. 1·00 if two coins turn both tails. *B* is paid Rs. 3·00 when the two coins do not match. Given the choice of being *A* or *B*, which one which one would you choose and what would be your strategy ? **[Delhi (M.B.A.) March 99]**

[**Ans.** Mixed strategies are $A\left(\frac{4}{5}, \frac{11}{15}\right)$; $B\left(\frac{4}{5}, \frac{1}{15}\right)$ and the expected value of the game is :

$$v = \{8p - 3(1-p)\}\, q + \{-3p + (1-p)\}(1-q) = \left\{8 \times \frac{4}{15} - 3 \times \frac{11}{15}\right\} \times \frac{4}{15} + \left\{-3 \times \frac{4}{15} + \frac{11}{15}\right\} \times \frac{11}{15} = -\frac{1}{15}]$$

15. Even though there are several manufacturers of scooters, two firms with branch names *Janata* and *Praja*, control their market in Western India. If both manufacturers make model changes of the same type for this market segment in the same year, their respective market shares remain constant. Likewise, if neither makes model changes, then also their market shares remain constant. The pay-off matrix in terms of increased decreased percentage market share under different possible conditions is given below :

		Praja		
		No change	Minor change	Major change
Janata	No change	0	-4	-10
	Minor change	3	0	5
	Major change	8	1	0

(i) Find the value of the game.

(ii) What change should Janata consider if this information is available only to itself ? **[Rajasthan (M. Com.) 98]**

[Ans. The optimum strategies for the two players are :

$$S_A = [\ 0\ ,\ 1/6\ ,\ 5/6\ ] \text{ and } S_B = [\ 0\ ,\ 5/6\ ,\ 1/6\ ]$$

(ii) *Janata* may consider to have minor change with probability 5/6 and of major change with probability 1/6.]

16. In a small town, there ar only two stores that handle sundry goods—*ABC* and *XYZ*. The total number of customers is equally divided between the reputation in the community, and they render equally good customer service. Assume that a gain of customers by *ABC* is a loss to *XYZ* and vice versa. Both stores plan to run annual pre-diwali sales during the first week of November. Sales are advertised through a *local newspaper, radio* and *television* media. With the aid of an advertising firm store *ABC* constrcted the game matrix given below. (Figures in the matrix represent a gain or loss of customers).

		Strategy of XYZ		
		Newspaper	Radio	Television
	Newspaper	30	40	− 80
Strategy of ABC	Radio	0	15	− 20
	Television	90	20	50

Determine optimum strategies and the worth of such strategies for both *ABC* and *XYZ*.

[A.I.M.A. (P.G. Dip in Management), Dec. 96]

17. Two breakfast food manufacturers, *ABC* and *XYZ* are competing for an increased market share. The pay-off matrix, shown in the following table, shows the increase in market share for *ABC* and decrease in market share of *XYZ*.

ABC	XYZ			
	Give coupons	Decrease price	Maintain present strategy	Increase advertising
Give coupons	2	− 2	4	1
Decrease price	6	1	12	3
Maintain present strategy	− 3	2	0	6
Increase advertising	2	− 3	7	1

Simplyfy the problem by the rule of dominance and find optimum strategies for both the manufacturers and value of the game. **[Delhi (M.B.A.) Dec. 95]**

[Ans. The optimum strategies for both the manufacturers are that manufacturer *ABC* should adopt strategy 'decrease price' 50% time and strategy maintain present strategy 50% time. Similarly, manufacturer *XYZ* should adopt strategy 'give coupons 10% times and strategy 'decrease price' 90% times. The value of the game will be in favour of manufacturer *ABC* and increase in market share would be 3·5.]

18. Two firms are competing for business under the conditions so that one firm's gain is another firm's loss. Firm A's pay-off matrix is given below :

		Firm B		
		No advertising	Medium advertising	Heavy advertising
	No advertising	10	5	− 2
Firm A	medium advertising	13	12	15
	Heavy advertising	16	14	10

Suggest optimum strategies for the two firms and the net outcome thereof. **[Delhi (M. Com.) 94]**

19. A steel company is negotiating with its union for revision of wages to its employees. The management, with the help of a mediator, has prepared a pay-off matrix shown below. Plus sign represents wages increase, while negative sign stands for wage decrease. Union has also constructed a table which is comparable to that developed by management. The management does not have the specific knowledge of game theory to select the best strategy or (strategies) for the firm. You have to assist the management on the problem. What game value and strategies are available to the opposing group ?

Additional ocst to Settl Co. (Rs.)
Union strategies

		U_1	U_2	U_3	U_4
	C_1	+ 2·50	+ 2·70	+ 3·50	− 0·20
Steel Co. strategies	C_2	+ 2·00	+ 1·60	+ 0·80	+ 0·80
	C_3	+ 1·40	+ 1·20	+ 1·50	+ 1·30
	C_4	+ 3·00	+ 1·40	+ 1·90	0

[Delhi (M.B.A.) Nov. 98]

[Hint. Since the company represents the 'minimizing player' and the union of the 'maximizing player', the given pay-off matrix is recast as follows by interchanging rows and columns.

$$\begin{array}{ll|cccc|}
 & & \multicolumn{4}{c}{\textbf{Company strategies}} \\
 & & C_1 & C_2 & C_3 & C_4 \\
\hline
 & U_1 & 2{\cdot}50 & 2{\cdot}00 & 1{\cdot}40 & 3{\cdot}00 \\
\textbf{Union strategies} & U_2 & 2{\cdot}70 & 1{\cdot}60 & 1{\cdot}20 & 1{\cdot}40 \\
 & U_3 & 3{\cdot}50 & 0{\cdot}80 & 1{\cdot}50 & 1{\cdot}90 \\
 & U_4 & -0{\cdot}20 & 0{\cdot}80 & 1{\cdot}30 & 0 \\
\hline
\end{array}$$

[Ans. Optimum strategy for the company is $\left(0, \frac{1}{13}, \frac{12}{13}, 0\right)$; for the union it is $\left(\frac{7}{13}, 0, \frac{6}{13}, 0\right)$ and the game value is $\frac{188}{1300}$ (representing increased wages).

20. Consider the game

$$\begin{array}{cc|ccc|}
 & & & \mathbf{B} & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 5 & 50 & 50 \\
\mathbf{A} & 2 & 1 & 1 & 0.1 \\
 & 3 & 10 & 1 & 10 \\
\hline
\end{array}$$

Verify that the strategies (1/6, 0, 5/6) for player A and (49/54, 5/54, 0) for player B are optimal and find the value of the game. [JNTU (B. Tech.) 2003]

9.15. GRAPHICAL METHOD FOR $(2 \times n)$ AND $(m \times 2)$ GAMES

The optimal strategies for a $(2 \times n)$ or $(m \times 2)$ matrix game can be located easily by a simple *graphical method*. This method enables us to reduce the $2 \times n$ or $m \times 2$ matrix game to 2×2 game that could be easily solved by the earlier methods.

If the graphical method is used for a particular problem, then the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. It is clear that if one player has only two strategies, the other will also use two strategies. Hence, graphical method can be used to find two strategies of the player. The method can be applied to $3 \times n$ or $m \times 3$ games also by carefully drawing three dimensional diagram.

9·15-1. Graphical Method for 2 × *n* Games

Consider the $(2 \times n)$ game, assuming that the game does not have a saddle point.

Since the player A has two strategies, it follows that $x_2 = 1 - x_1$, $x_1 \geq 0$, $x_2 \geq 0$. Thus, for each of the pure strategies available to the player B, the expected payoff for the player A, would be as follows :

Table 9·27

$$\begin{array}{ccc|ccccc|}
 & & & & & \mathrm{B} & & \\
 & & & y_1 & y_2 & y_3 & \dots & y_n \\
 & & & B_1 & B_2 & B_3 & \dots & B_n \\
\hline
\mathrm{A} & x_1 & A_1 & v_{11} & v_{12} & v_{13} & \dots & v_{1n} \\
 & 1 - x_1 & A_2 & v_{21} & v_{22} & v_{23} & \dots & v_{2n} \\
\hline
\end{array}$$

This shows that the player A's expected payoff varies linearly with x_1.

Table 9·28

B's Pure Strategies	A's Expected Payoff $E_j(x_1)$
B_1	$v_{11}x_1 + v_{21}(1 - x_1) = (v_{11} - v_{21})x_1 + v_{21}$
B_2	$v_{12}x_1 + v_{22}(1 - x_1) = (v_{12} - v_{22})x_1 + v_{22}$
$\vdots$	$\vdots \quad \vdots \quad \vdots$
B_n	$v_{1n}x_1 + v_{2n}(1 - x_1) = (v_{1n} - v_{2n})x_1 + v_{2n}$

According to the maximin criterion for mixed strategy games; the player A should select the value of x_1 so as to maximize his minimum expected payoff. This may be done by plotting the following straight lines :

$$E_1(x_1) = (v_{11} - v_{21})\, x_1 + v_{21}$$
$$E_2(x_1) = (v_{12} - v_{22})\, x_1 + v_{22}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$E_n(x_1) = (v_{1n} - v_{2n})\, x_1 + v_{2n}$$

as functions of x_1. The lowest boundary of these lines will give the minimum expected payoff as function of x_1. The highest point on this lowest boundary would then give the *maximin* expected payoff and the optimum value of x_1 $(= x_1{}^*)$.

Now determine only two strategies for player B corresponding to those two lines which pass through the *maximin* point P (Fig. 9·1). This way, it is possible to reduce the game to 2×2 which can be easily solved either by using formulae given in Sec. 9·13 or by *arithmetic method*.

Outlines of Graphical Method :

To determine maximin value $\underline{v}$, we take different values of x_1 on the horizontal line and values of $E(x_1)$ on the vertical axis. Since $0 \le x_1 \le 1$, the straight line $E_j(x_1)$ must pass through the points $\{0, E_j(0)\}$ and $\{1, E_j(1)\}$, where $E_j(0) = v_{2j}$ and $E_j(1) = v_{1j}$. Thus the lines $E_j(x_1) = (v_{1j} - v_{2j})\, x_1 + v_{2j}$ for $j = 1, 2, ..., n$ can be drawn as follows :

Step 1. Construct two vertical axes, *axis* 1 at the point $x_1 = 0$ and *axis* 2 at the point $x_1 = 1$.

Step 2. Represent the payoffs $v_{2j}, j = 1, 2, ..., n$ on *axis* 1 and payoff $v_{1j}, j = 1, 2, ..., n$ on *axis* 2.

Step 3. Join the point representing v_{1j} on *Axis* 2 to the point representing v_{2j} on *axis* 1. The resulting straightline is the expected payoff line $E_j(x_1), j = 1, 2, ..., n$.

Step 4. Mark the lowest boundary of the lines $E_j(x_1)$ so plotted, by *thickline* segments. The *highest point* on this *lowest boundary* gives the *maximin* point *P* and identifies the two critical moves of player *B*.

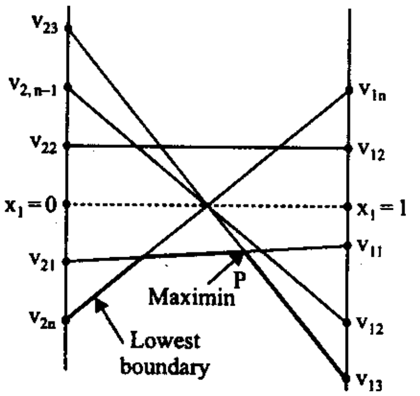


Fig. 9.1 Graphical solution of 2xn games.

If there are more than two lines passing through the maximin point *P*, there are ties for the optimum mixed strategies for player *B*. Thus any two such lines with *opposite* sign slopes will define an alternative optimum for *B*.

9·15-2. Graphical Solution of $m \times 2$ Games

The $(m \times 2)$ games are also treated in the like manner except that the *minimax* point *P* is the *lowest point* on the *uppermost boundary* instead of highest point on the lowest boundary.

From this discussion, it is concluded that any $(2 \times n)$ or $(m \times 2)$ game is basically equivalent to a (2×2) game.

Now each point of the discussion is explained by solving numerical examples for $(2 \times n)$ and $(m \times 2)$ games.

Q. Explain the graphical method of solving $(2 \times n)$ and $(m \times 2)$ games.

Example 13. *Solve the following* (2×3) *game graphically.*

Table 9·29

			y_1	y_2	y_3
			I	II	III
A	x_1	I	1	3	11
	$1-x_1$	II	8	5	2

[JNTU (Mech. & Prod.) 2004, 03, 02; Agra 99; Delhi B.Sc. (Math) 91]

Solution. This game does not have a saddle point. Thus the player *A*'s expected payoff corresponding to the player *B*'s pure strategies are given (*Table 9·30*).

Three expected payoff lines are :

$$E(x_1) = -7x_1 + 8,\ E(x_1) = -2x_1 + 5 \text{ and } E(x_1) = 9x_1 + 2$$

and can be plotted on a graph as follows [see Fig. 9·2]

Table 9·30

B's Pure Strategies	*A*'s Expected Payoff $E(X_1)$
I	$E(x_1) = 1.x_1 + 8(1 - x_1) = -7x_1 + 8$
II	$E(x_1) = 3.x_1 + 5.(1 - x_1) = -2x_1 + 5$
III	$E(x_1) = 11x_1 + 2(1 - x_1) = 9x_1 + 2$

First, draw two parallel lines one unit apart and mark a scale on each. These two lines will represent two strategies available to the player *A*. Then draw lines to represent each of player *B*'s strategies.

For example, to represent the player B's Ist strategy, join mark 1 on scale I to mark 8 on scale II; to represent the player B's second strategy, join mark 3 on scale I to mark 5 on scale II, and so on. Since the expected payoff $E(x_1)$ is the function of x_1 alone, these three expected payoff lines can be drawn by taking x_1 as x-axis and $E(x_1)$ as y- axis.

Points A, P, B, C on the lowest boundary (shown by a thick line in Fig. 9·2) represent the lowest possible expected gain to the player A for any value of x_1 between 0 and 1. According to the *maximin* criterion, the player A chooses the best of these worst outcomes.

Clearly, the *highest* point P on the *lowest* boundary will give the largest expected gain PN to A. So best strategies for the player B are those which pass through the point P. Thus, the game is reduced to 2 × 2 (*Table 9·31*).

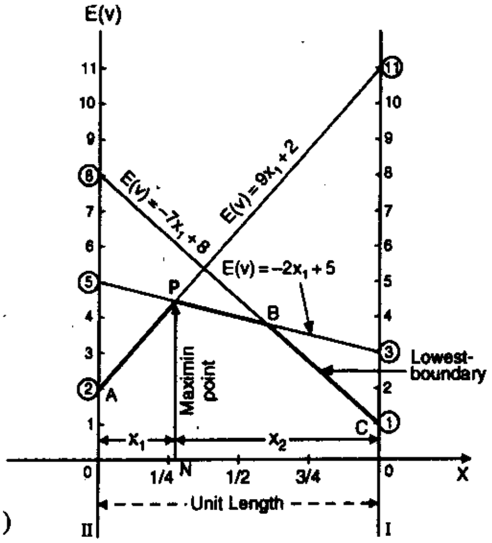


Fig. 9.2 Graphical representation for solving (2 × *n*) game.

Now, by solving the simultaneous equations

$3x_1 + 5x_2 = v,\ 11x_1 + 2x_2 = v,\ x_1 + x_2 = 1$ (*For player A*)

$3y_2 + 11y_3 = v,\ 5y_2 + 2y_3 = v,\ y_2 + y_3 = 1$ (*For player B*)

the solution of the game is obtained as follows :

(*i*) The player A chooses the optimal mixed strategy $(x_1, x_2) = (3/11,\ 8/11)$,

(*ii*) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3) = (0, 2/11, 9/11)$,

(*iii*) The value of the game to the player A is $v = 49/11$.

Table 9·31

		B	
		II	III
A	I	3	11
	II	5	2

Example 14. *Solve the game graphically whose payoff matrix for the player A is given in Table 9·32 :*

Table 9·32

		B	
		I	II
	I	2	4
A	II	2	3
	III	3	2
	IV	−2	6

Solution. The game does not have a saddle point. Let y_1 and y_2 (= $1 - y_1$) be mixed strategies of the player B.

The four straight lines thus obtained are :

$$E(y_1) = -2y_1 + 4,\ E(y_1) = -y_1 + 3,$$

$$E(y_1) = y_1 + 2,\ E(y_1) = -8y_1 + 6,$$

and these are plotted in Fig. 9·3. In this case, the *minimax* point is determined as the *lowest point P* on the uppermost boundary. Lines intersecting at the minimax point P correspond to the player A's pure strategies I and III. This indicates $x_2 = x_4 = 0$. Thus, the reduced game is given in *Table 9·34*.

Table 9·33

A's Pure Strategies	B's Expected Payoff $E(y_1)$
I	$E(y_1) = 2y_1 + 4(1 - y_1)$
II	$E(y_1) = 2y_1 + 3(1 - y_1)$
III	$E(y_1) = 3y_1 + 2(1 - y_1)$
IV	$E(y_1) = -2y_1 + 6(1 - y_1)$

Now, solve this (2 × 2) game by solving the simultaneous equations :

$2x_1 + 3x_3 = v, 4x_1 + 2x_3 = v, x_1 + x_3 = 1$ (*For A*)

$2y_1 + 4y_2 = v, 3y_1 + 2y_2 = v, y_1 + y_2 = 1$ (*For B*)

to get the solution :

(*i*) The player A chooses the optimal mixed strategy, $(x_1, x_2, x_3, x_4) = (1/3, 0, 2/3, 0)$.

(*ii*) The player B chooses the optimal mixed strategy, $(y_1, y_2) = (2/3, 1/3)$.

(*iii*) The value of the game to the player A is $v = 8/3$.

Table 9·34

		B	
		I	II
A	I	2	4
	III	3	2

Remark. If there are more than two lines passing through the maximin (minimax) point P, this would imply that there are many ties for optimal mixed strategies for the player B. Thus, any two lines having opposite signs for their slops will define an alternative optimum solutions.

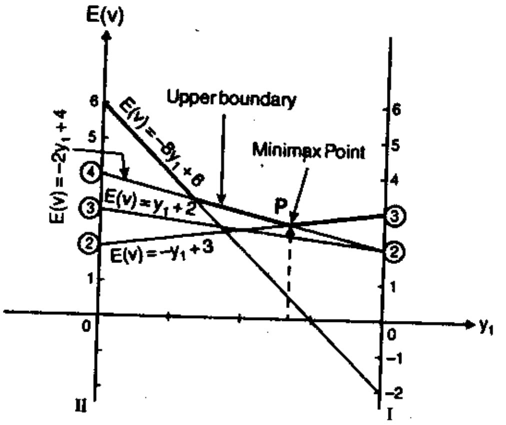


Fig. 9.3. Graphical representation for $(m \times 2)$ game.

9·15-3. Illustrative Examples

Example 15. *Solve the following* (2×4) *game.*

		B			
		I	II	III	IV
A	I	2	2	3	-1
	II	4	3	2	6

[Agra M.Sc. (Maths.) 99; Meerut M.Sc. (Math.) 99, 96]

Solution. This game does not have saddle point. Thus, the player A's expected payoffs corresponding to the player B's pure strategies are given below (*Table 9·35*) :

Table 9·35

B's Pure Strategies	*A*'s Expected Payoff $E(x_1)$
I	$E(x_1) = -2x_1 + 4$
II	$E(x_1) = -x_1 + 3$
III	$E(x_1) = x_1 + 2$
IV	$E(x_1) = -7x_1 + 6$

These four straight lines are then plotted in Fig. 9.4.

It follows from *Fig.* 9·4 that maximin occurs at $x_1 = 1/2$. This is the point of intersection of *any* two of the lines joining (2) to (3); (3) to (2); (6) to (− 1). As mentioned in the above remark, any *two lines having opposite signs for their slopes will define an alternative* optimum solution. The combination of lines $E(x_1) = -x_1 + 3$ and $E(x_1) = -7x_1 + 6$ must be excluded as being non- optimal. So the game can be reduced to (2×2) in the following manner.

It is also important to note that the average of above two payoff matrices (*Table 9·36* and *19·37*) will also be the additional possibility of reducing the game to (2×2). Thus, the additional possibility of (2×2) game will also yield a new optimal solution which mixes three strategies II, III and IV. Then (2×2) game is solved by solving the governing simultaneous equations.

Table 9·36
1st possibility

		B	
		II	III
A	I	2	3
	II	3	2

Table 9·37
2nd possibility

		B	
		III	IV
A	I	3	-1
	II	2	6

Table 9·38
Additional possibility

		B	
A	I	$\frac{2+3}{2} = 5/2$	$\frac{3-1}{2} = 1$
	II	$\frac{3+2}{2} = 5/2$	$\frac{2+6}{2} = 4$

The first possibility of the solution of (2×4) game with reduced (2×2) matrix is :

(*i*) The player A chooses the optimal mixed strategy $(x_1, x_2) = (1/2, 1/2)$.

(*ii*) The player B chooses the optimal mixed strategy $(y_1, y_2, y_3, y_4) = (0, 1/2, 1/2, 0)$.

(*iii*) The value of the game to the player A is $2\frac{1}{2}$.

Similarly, solution of the game with reduced matrix in 2nd possibility is :

(*i*) The player A chooses the optimal mixed strategy $(1/2, 1/2)$.

(*ii*) The player B chooses the optimal mixed strategy $(0, 0, 7/8, 1/8)$.

(*iii*) The value of the game to the player A is $2\frac{1}{2}$.

Solution of the game with reduced matrix in additional possibility can be obtained easily, because it has a saddle point $5/2$. So, the value of the game to the player A is $5/2$. It has been observed that the formulae in *Sec. 9·13* will yield an incorrect solution in this case.

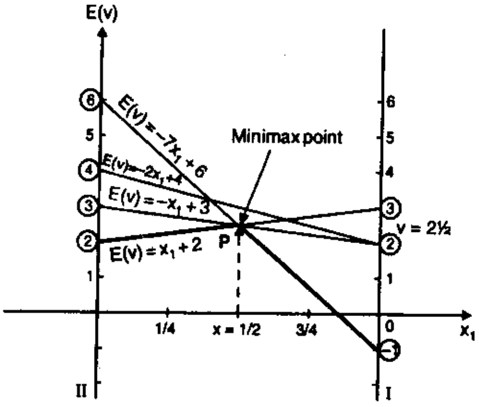


Fig. 9.4

Example 16. *Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs. 400 per colour set and Rs. 300 per black & white set. Firm B can, on the other hand, make either 300 colour sets, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a partiuclar type of set.*

Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game. **[Bombay (M.M.S.) 97]**

Solution. For firm A, the strategies are :

A_1 : make 150 colour sets,

A_2 : make 150 black & white sets.

For firm B, the strategies are :

B_1 : make 300 colour sets,

B_2 : make 150 colour and 150 black & white sets,

B_3 : make 300 black and white sets.

For the combination A_1B_1, the profit to firm A would be : $\dfrac{150}{150 + 300} \times 150 \times 400 \;=\;$ Rs. 20,000

wherein $(150/150 + 300)$ represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar way, other profit figures may be obtained as shown in the following pay-off matrix :

$$\begin{array}{ccc} & & B\text{'s strategy} & & \\ & & B_1 & B_2 & B_3 \\ A\text{'s strategy} & A_1 & 20{,}000 & 30{,}000 & 60{,}000 \\ & A_2 & 45{,}000 & 45{,}000 & 30{,}000 \end{array}$$

Since no saddle point exists, we shall determine optimum mixed strategy. The data are plotted on graph as shown in the adjoining Fig. 9·5 :

Lines joining the pay-offs on axis I with the pay-offs on axis II represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point P on the lower envelope of A's expected pay-off equation. This point P represents the maximin expected value of the game for firm A. The lines B_1 and B_3 passing through P, define the relevant moves B_1 and B_3 that alone from B needs to adopt. The solution to the original 2×3 game, therefore, reduces to that of the simple game with 2×2 pay-off matrix as follows :

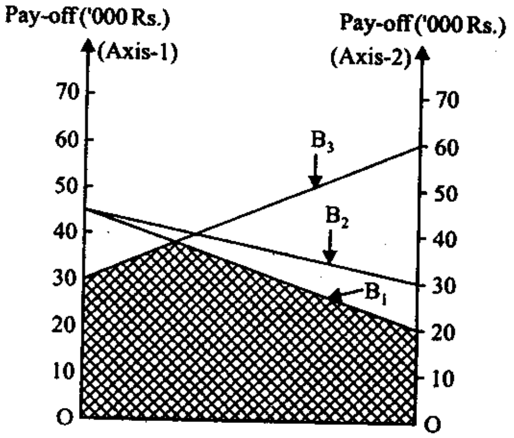


Fig. 9.5 : Graphic solution to the Game

$$\begin{array}{lccc} & & \multicolumn{2}{c}{B\text{'s strategy}} \\ & & B_1 & B_2 \\ A\text{'s strategy} & A_1 & \left[\begin{matrix} 20{,}000 \\ 45{,}000 \end{matrix}\right. & \left.\begin{matrix} 60{,}000 \\ 30{,}000 \end{matrix}\right] \\ & A_2 & & \end{array}$$

Correspondingly,

$$p_1 = \frac{a_{22} + a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{3}{11}$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{30{,}000 - 60{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} = \frac{6}{11}$$

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{21}) - (a_{12} + a_{21})} = \frac{20{,}000 \times 30{,}000 - 60{,}000 \times 45{,}000}{(20{,}000 + 30{,}000) - (60{,}000 + 45{,}000)} \simeq 38{,}182$$

Example 17. *Solve the following game by graphical method.*

		B			
		y_1	y_2	y_3	y_4
A	x_1	19	6	7	5
	x_2	7	3	14	6
	x_3	12	8	18	4
	x_4	8	7	13	−1

[VTU (BE Mech.) 2002]

Solution.

Step 1. The first step is to search for a saddle point. There is no saddle point in this problem.

Step 2. The second step is to observe if the game can be reduced by dominance. Since all the cell entries in column 2 are less than the corresponding values in columns 1 and 3, hence columns 1 and 3 are dominated by column 2 and thus the reduced matrix becomes as shown on the right.

		y_2	y_4
A	x_1	6	5
	x_2	3	6
	x_3	8	4
	x_4	7	−1

Again, since all the cell entries for row 3 are more than those for row 4, hence row 3 dominates row 4 and the matrix is thus reduced to the following (3 × 2) game.

		B	
		y_2	$y_4 = 1 - y_2$
A	x_1	6	5
	x_2	3	6
	x_4	8	4

Step 3. Now the above (3 × 2) matrix game can be solved by graphical method proceeding as in *Example 14*.

We can, thus, immediately reduce the (3 × 2) game to the following (2 × 2) game which can be easily solved by arithmetic method (since it has no saddle point).

		B			
		II	IV		
A	I	6	5	3	3/4
	II	3	6	1	1/4
		1	3		
		1/4	3/4		

Therefore, optimal strategies are :

(i) A (3/4, 1/4, 0, 0), *(ii)* B (0, 1/4, 0, 3/4) *(iii)* Value of the game, $v = \frac{6 \times 1 + 3 \times 6}{1 + 3} = 5\frac{1}{4}$

9·15-4 Method of Subgames for (2 × *n*) or (*m* × 2) Games

To explain the method we consider the following interesting example.

Example 18. *Two airlines operate the same air-route, both trying to get as large a market as possible. Based on a certain market, daily gains and losses in rupees are shown in table below, in which positive values favour airline A and negative values favour airline B. Find the solution for the game.*

			Airline B		
			Does nothing	Advertises special rates	Advertises special features (*i.e.* movies, fine food)
			1	2	3
Airline A	Advertises special rates	1	275	−50	−75
	Advertises special features (*i.e.* movies, fine food)	2	125	130	150

Solution.

***Step.* 1.** First observe that the game neither has a saddle point, nor it can be reduced by dominance. This game can be solved by algebraic method as described in [Sec. 9·17], but we shall solve this game here by the *method of subgames.*

***Step.* 2.** This (2 × 3) game can be considered as three (2 × 2) subgames :

Subgame I

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & 1 & 2 \\ \hline A & 1 & 275 & -50 \\ & 2 & 125 & 130 \\ \hline \end{array}$$

(Deleting col. 3)

Subgame II

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & 1 & 3 \\ \hline A & 1 & 275 & -75 \\ & 2 & 125 & 150 \\ \hline \end{array}$$

(Deleting col. 2)

Subgame III

$$\begin{array}{cc|cc|} & & \multicolumn{2}{c}{B} \\ & & 2 & 3 \\ \hline A & 1 & -50 & -75 \\ & 2 & 130 & 150 \\ \hline \end{array}$$

(Deleting col. 1)

***Step* 3.** It is observed that airline *B* which has more number of columns (than the number of rows for *A*), has more flexibility, generally resulting in a better strategy. In order to find optimum strategy for airline *B*, all the above three (2 × 2) subgames must be solved. We solve them below by *Arithmetic Method.*

Subgame 1. ***(No saddle point)*** **:**

(i) The strategy for *A* is, (1/66, 65/66)

(ii) The strategy for *B* is, (36/66, 30/66, 0)

(iii) Value of the game = Rs. $\frac{275 \times 1 + 125 \times 65}{66}$ = Rs. 127·30.

$$\begin{array}{cc|c|c|ccc} & & \multicolumn{2}{c}{B} & & & \\ & & 1 & 2 & & & \\ \hline A & 1 & 275 & -50 & 5 & 1 & 1/66 \\ & 2 & 125 & 130 & 325 & 65 & 65/66 \\ \hline & & 180 & 150 & & & \\ & \text{or} & 6 & 6 & & & \\ & \text{or} & 36 & 30 & & & \\ & \text{or} & 36/66 & 30/66 & & & \end{array}$$

Subgame 2. ***(No saddle point)*** **:**

(i) The strategy for *A* is : (1/15 , 14/15)

(ii) The strategy for *B* is : (9/15 , 0, 6/15)

(iii) The value of the game, v = Rs. $\frac{275 \times 3 - 75 \times 2}{5}$ = Rs. 135.

$$\begin{array}{cc|c|c|ccc} & & \multicolumn{2}{c}{B} & & & \\ & & 1 & 3 & & & \\ \hline A & 1 & 275 & -75 & 25 & 1 & 1/15 \\ & 2 & 125 & 150 & 350 & 14 & 14/15 \\ \hline & & 225 & 150 & & & \\ & & 3 & 2 & & & \\ & & 3/5 & 2/5 & & & \\ & & 9/15 & 6/15 & & & \end{array}$$

Subgame 3. ***(Has a saddle point)*** **:**

Thus, this game has a saddle point (2, 2). So the solution is *A* (0, 1), *B* (0, 1, 0), v = Rs. 130.

$$\begin{array}{cc|c|c|c} & & \multicolumn{2}{c}{B} & \\ & & 2 & 3 & \text{Row Min.} \\ \hline A & 1 & -50 & -75 & -75 \\ & 2 & 130^* & 150 & 130^* \\ \hline \text{Col. Max.} & & 130^* & 150 & \end{array}$$

Now since airline *B* has the flexibility to play any two out of the available courses of action, it will play those strategies for which the loss occurring to the airline is minimum. Since all the values for the subgames are positive, the airline *A* is the winner. Hence airline *B* will play subgame 1 for which the loss is minimum, *i.e.* Rs. 127·30. Hence the complete solution to the problem is :

Optimum strategies : *A* (1/66 , 65/66), *B* (36/66 , 30/66, 0)

Value of the game, v = 127·30

Note Carefully. Here subgame 3 has a saddle point, hence arithmetic method should not be applied to solve it. If it is applied, the resulting solution will be incorrect.

EXAMINATION PROBLEMS

Use graphical method to reduce the following games and hence solve :

1.
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

[Ans. *A* (4/11, 7/11), *B* (0, 7/11, 4/11), $v = -5/11$]

2.
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix} \end{array}$$

[Ans. (3/7, 4/7), (2/7, 0, 5/7), $v = 8/7$]

3.
$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 2 & 7 \\ 3 & 5 \\ 11 & 2 \end{bmatrix} \end{array}$$

[Ans. (9/14, 0, 5/14), (5/14, 9/14), $v = 73/14$]

4.
$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 1 & 2 \\ 5 & 6 \\ -7 & -9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix} \end{array}$$

[Meerut M.Sc (Math.) 98 BP, 96.]

[Hint. Upper boundary is the line (5 → 6)]

[Ans. (0, 1, 0, 0); (1, 0), $v = 5$]

5. Solve the game whose payoff matrix is given below.

(i)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & 2 & 7 \\ 1 & 9 & 3 \end{bmatrix} \end{array}$$

[Meerut 99]

(ii)
$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -3 \end{bmatrix} \end{array}$$

(iii)
$$\begin{bmatrix} -4 & 3 & -1 \\ 6 & -4 & -2 \end{bmatrix}$$

6. Solve the following 2 × 4 game graphically :

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{array}{ccccc} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 2 & 1 & 0 & -2 \\ A_2 & 1 & 0 & 3 & 3 \end{array} \end{array}$$

[Ans. (2/5, 3/5), (0, 4/5, 0, 1/5), $v = 2/5$] **[Agra 92; Madurai B.Sc. (Comp. Sc.) 92, Jadavpur M.Sc. (Math) 92]**

7. Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose payoff matrix is given as follows :

$$\begin{array}{cc} & \textbf{Player } A \\ \textbf{Player } B & \begin{bmatrix} 1 & 3 & -1 & 4 & 2 & -5 \\ -3 & 5 & 6 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

[Ans. (0, 3/5, 0, 2/5, 0, 0), (4/5, 1/5), $v = 17/5$]

8. Two companies *A* and *B* are competing for the same product. Their different strategies are given in the following payoff matrix :

(i)
$$\begin{array}{cc} & \textbf{Company } A \\ \textbf{Company } B & \begin{array}{cccc} & A_1 & A_2 & A_3 \\ B_1 & 2 & -2 & 3 \\ B_2 & -3 & 5 & -1 \end{array} \end{array}$$

(ii)
$$\begin{array}{cc} & \textbf{Company } B \\ \textbf{Company } A & \begin{bmatrix} 4 & -3 & 3 \\ -3 & 1 & -1 \end{bmatrix} \end{array}$$

What are the best strategies for both the companies ? **[Hint.** Here III column is dominated by 1/2 (I + II) column.]
Find out the value of the game.

[Ans. (0, 4/11, 7/11), (6/11, 5/11), $v = 13/11$]

[Ans. *A* (4/11, 7/11), *B* (4/11, 7/11, 0), $v = -5/11$]

9. Solve the following zero-sum games where player *B* plays with player *A*.

(i)
$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} 2 & -4 & 6 & -3 & 5 \\ -3 & 4 & -4 & 1 & 0 \end{bmatrix} \end{array}$$

[JNTU (B. Tech.) 2003]

(ii)
$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{bmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{bmatrix} \end{array}$$

[VTU 2002; Agra 99; IAS (Main) 98]

[Ans. (i) *A* (4/9, 5/9), *B* (4/9, 0, 5/9, 0), $v = -7/9$. (ii) *A* (0, 0, 0, 13/20, 7/20), *B* (11/20, 9/20), $v = 23/20$]

10. Obtain the optimal strategies for both persons and the value of the game for two person zero-sum game whose payoff matrix is as follows :

$$\begin{array}{cc} & \textbf{Player } B \\ \textbf{Player } A & \begin{array}{ccc} & B_1 & B_2 \\ A_1 & 1 & -3 \\ A_2 & 3 & 5 \\ A_3 & -1 & 6 \\ A_4 & 4 & 1 \\ A_5 & 2 & 2 \\ A_6 & -5 & 0 \end{array} \end{array}$$

[Ans. (0, 3/5, 0, 2/5, 0, 0), (4/5, 1/5), $v = 17/5$]

11. Solve graphically the following games :

(a) $\begin{bmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & -2 & -3 \\ 2 & 1 & 4 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 19 & 15 & 7 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 & 3 & -2 \\ 4 & 3 & 2 & 6 \end{bmatrix}$

[Delhi (B.Sc.) 90] **[Delhi (B.Sc.) 93]**

[**Ans.** (a) (3/4, 1/4), (3/4, 0, 0, 1/4), $v = 1/2$; (b) (1/4, 3/4), (1/2, 0, 1/2, 0), $v = 1/2$;
(c) (15/16, 1/16), (0, 11/16, 0, 5/16), $v = 245/16$; (d) (4/9, 5/9), (0, 0, 8/9, 1/9), $v = 22/9$]

12. Solve the following game using the graphical method :

$$\begin{array}{ccc} & & \textbf{\textit{B}'s strategy} \\ & & \begin{array}{cc} B_1 & B_2 \end{array} \\ & \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{bmatrix} \end{array}$$

$$\text{A's strategy}$$

[JNTU (B. Tech.) 2003 (Type); Gjuarat (M.B.A.) 98]

13. The following matrix represents the pay-off to A in a rectangular game between A and B :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 19 & 15 & -5 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

By the notion of dominance show that the game is equvalent to one represented by a 2 × 4 matrix which is a submatrix of the above matrix. Then obtain a solution of the game graphically. **[Meerut 2002]**

9.16. MATRIX METHOD FOR $M \times N$ GAMES

A convenient and systematic procedure of finding all solutions of $m \times n$ reactangular games can be provided by following theorem.

Theorem 9·6. *Let* **A** *be the payoff matrix of an* $m \times n$ *rectangular game. Let the strategies of the row player be given by an m-tuple* $\mathbf{x} \in S_m$ *and those of the column player by an n-tuple* $y \in S_n$, *where* S_m , S_n *are the set of strategies for the two players. Then a necessary and sufficient condition that* $\mathbf{x}_0 \in S_m$ *and* $\mathbf{y}_0 \in S_n$ *be solution of the game is that there exists a square submatrix* **B** *of* **A** *of order r, such that* $\mathbf{u}_r^T$ *(adj* **B**) $\mathbf{u}_r \neq 0$, and that

$$v = \frac{|\mathbf{B}|}{\Delta}, \quad p_*^T = \frac{\mathbf{u}_r^T \, adj \, \mathbf{B}}{\Delta}, \quad q_*^T = \frac{\mathbf{u}_r^T \, (adj \, \mathbf{B})^T}{\Delta},$$

where $\Delta = \mathbf{u}_r^T \, (adj \, \mathbf{B}) \mathbf{u}_r$ and $\mathbf{x}^*$ $(\mathbf{y}^*)$ *are obtained from* $\mathbf{x}_0$ $(\mathbf{y}_0)$ *by deleting the elements of the corresponding rows (columns) to obtain* **B** *from* **A** *and* $\mathbf{u}_r$ *is the column vector (1, 1, ... 1) having r components.*

The proof of this theorem is beyond the scope of the book.

Outlines of Matrix Method for $m \times n$ games :

The systematic matrix method may be summarized as follows :

Step **1.** Given an $m \times n$ payoff matrix **A**, determine all square submatrices **B** of **A**. (order of **B** is greater than 1)

Step **2.** For a submatrix **B** of order r $(r \geq 2)$, determine the quantities v , $\mathbf{x}^*$, $\mathbf{y}^*$ as given by above *Theorem 9·6.*

Step **3.** Check whether $\mathbf{x}^*, \mathbf{y}^* \in S_r$ *i.e.*, check whether each of them consists of non-negative components that sum up to unity. If not, reject this **B** and go back to step 2 with another submatrix **B**.

Step **4.** If $\mathbf{x}^*$, $\mathbf{y}^* \in S_r$, form $\mathbf{x}_0$, $\mathbf{y}_0$ by adding relevant zero components to $\mathbf{x}^*$, $\mathbf{y}^*$.

Step **5.** Test whether the solution $(v, \mathbf{x}_0 , \mathbf{y}_0)$ thus obtained is admissible. This can be done by verifying that

$$E(\mathbf{x}_0 , \mathbf{e}_j) = \sum_{i=1}^{m} x_{0i} \, v_{ij} \geq v , j = 1, 2, ..., n \text{ and } E(\mathbf{e}_i , \mathbf{y}_0) = \sum_{j=1}^{n} y_{0j} \, v_{ij} \leq v , i = 1, 2, ... m.$$

If the above solution $(v, \mathbf{x}_0 , \mathbf{y}_0)$ is not admissible, reject the current **B** .

Step **6.** Select another submatrix **B** and then to to step 2.

Note. Since the fundamental theorem ensures that every rectangular game has a solution, the above procedure will provide us at least one admissible solution.

The method can be easily understood by the following numerical examples.

Example 19. *Use matrix method to solve a game whose payoff matrix is :* $\begin{pmatrix} 3 & 6 \\ 5 & 5 \\ 9 & 3 \end{pmatrix}$

Solution.

***Step* 1.** First observe that the matrix has no saddle point.

***Step* 2.** Then consider the following three 2 × 2 square submatrices :

$$\mathbf{B} = \begin{pmatrix} 3 & 6 \\ 5 & 5 \end{pmatrix}\; \mathbf{C} = \begin{pmatrix} 5 & 5 \\ 9 & 3 \end{pmatrix}, \text{ and } \mathbf{D} = \begin{pmatrix} 3 & 6 \\ 9 & 3 \end{pmatrix}.$$

***Step* 3.** Now starting with submatrix B, we compute the following :

$$|\mathbf{B}| = -15, \quad adj\,\mathbf{B} = \begin{pmatrix} 5 & -6 \\ -5 & 3 \end{pmatrix} \text{ so that } (adj\,\mathbf{B})^T = \begin{pmatrix} 5 & -5 \\ -6 & 3 \end{pmatrix}$$

Thus we get

$$\mathbf{u}_2^T\,(adj\,\mathbf{B}) = (1, 1)\begin{pmatrix} 5 & -6 \\ -5 & 3 \end{pmatrix} = (0, -3) \quad \mathbf{u}_2^T\,(adj\,\mathbf{B})^T = (1, 1)\begin{pmatrix} 5 & -5 \\ -6 & 3 \end{pmatrix} = (-1, 2)$$

and $\quad \Delta = \mathbf{u}_2^T\,(adj\,\mathbf{B})\,\mathbf{u}_2 = (0, -3)\,[1, 1] = -3\,.$

***Step* 4.** Substituting above values in the formulae of *Theorem 9.6,* we get

$$v = \frac{|\mathbf{B}|}{\Delta} = \frac{-15}{-3} = 5, \quad \mathbf{x}^* = \mathbf{u}_2^T\,adj\,\frac{|\mathbf{B}|}{\Delta} = -\frac{1}{3}\,(0, 3) = (0, 1)$$

$$\mathbf{y}^* = \mathbf{u}_2^T\,(adj\,\mathbf{B})^T/\Delta = -\frac{1}{3}\,(-1, -2) = (1/3, 2/3)\,.$$

Clearly, $\mathbf{x}^*, \mathbf{y}^* \in S_2$. Also since third row (and no column) was deleted from **A** in order to get the submatrix **B**, we have $\mathbf{x}_0^T = (0, 1, 0),\ \mathbf{y}_0^T = (1/3, 2/3)$.

***Step* 5.** Now test the solution $(v, \mathbf{x}_0, \mathbf{y}_0)$ for optimality, by checking

$E(\mathbf{x}_0, \mathbf{e}_1) = 0 \times 3 + 1 \times 5 + 0 \times 9 = 5 = v, \quad E(\mathbf{x}_0, \mathbf{e}_2) = 0 \times 6 + 1 \times 5 + 0 \times 3 = 5 = v,$

$E(\mathbf{e}_1, \mathbf{y}_0) = (1/3) \times 3 + 2/3 \times 6 = 5 = v, \quad E(\mathbf{e}_2, \mathbf{y}_0) = (1/3) \times 9 + 2/3 \times 3 = 5 = v\,\cdot$

Above results show that the current $(v, \mathbf{x}_0, \mathbf{y}_0)$ is an admissible solution.

***Step* 6.** Next repeat *steps 3 to 5* starting with the submatrix C to get the admissible solution $v = 5, \mathbf{x}_0^T = (0, 1, 0)$, and $\mathbf{y}_0^T = (1/3, 2/3)$.

***Step* 7.** Again, repeat steps 3 to 5 starting with the submatrix **D** to obtain the admissible solution

$$v = 5, \quad \mathbf{x}_0^T = (2/3, 0, 1/3), \quad \mathbf{y}_0^T = (1/3, 2/3)\,.$$

***Step* 8.** Finally, we observe that the given problem has two solutions :

(i) $\mathbf{x}_0^T = (0, 1, 0), \mathbf{y}_0^T = (1/3, 2/3)\,;\ v = 5$ *(ii)* $\mathbf{x}_0^T = (2/3, 0, 1/3), \mathbf{y}_0^T = (1/3, 2/3)\,;\ v = 5.$

Thus the column player has unique optimal startegy $\mathbf{y}_0^T = (1/3, 2/3)$ while the row player has infinite number of optimal strategies given by the linear combination

$$\mathbf{x}_0^T = \lambda(0, 1, 0) + (1 - \lambda)\,(2/3, 0, 1/3), 0 \le \lambda \le 1.$$

Example 20. *Use matrix method to solve the game whose payoff matrix is*

$$A = \begin{pmatrix} -1 & 3 & -3 \\ 2 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

Solution. Here we find that

$$adj\,\mathbf{A} = \begin{pmatrix} -3 & -3 & 9 \\ 6 & 6 & -3 \\ 2 & 7 & -6 \end{pmatrix}$$

which gives the admissible solution, $\mathbf{x}_0^T = (1/3, 2/3, 0), \mathbf{y}_0^T = (1/5, 3/5, 1/5)\,;\ v = 1.$

It is observed that nine 2 × 2 submatrices of **A** are possible. But only one of them, *viz.*, $\mathbf{B} = \begin{pmatrix} 3 & -3 \\ 0 & 3 \end{pmatrix}$ provides an additional admissible solution : $\mathbf{x}_0^T = (1/3, 2/3, 0),\ \mathbf{y}_0^T = (0, 2/3, 1/3);\ v = 1.$

Thus here the row player has a unique optimal strategy $\mathbf{x}_0^T = (1/3, 2/3, 0)$, while the column player has infinite optimum strategies given by $\mathbf{y}_0^T = \lambda\,(1/5, 3/5, 1/5) + (1 - \lambda)\,(0, 2/3, 1/3), 0 \le \lambda \le 1$

The value of the game is 1.

From above two examples we observe that the matrix method described above is highly systematic but involves laborious computations in the case of large payoff matrices. For example, the number of square submatrices of a matrix of higher order is itself a very large number and so it becomes exceedingly tiresome to find the adjoint of a square matric of higher order. Thus in order to remove this difficulty a short-cut matrix method for $n \times n$ games is developed.

9·16-1. A Short-Cut Matrix Method for $n \times n$ Games

A short-cut matrix method is described below to solve any $n \times n$ games quite efficiently, though it provides only *one* optimal solution.

***Step* 1.** Let $A = \{v_{ij}\}$ be the $n \times n$ payoff matrix. Obtain a new matrix C whose first column is obtained from A by subtracting its 2nd column from 1st; second column is obtained by subtracting A's 3rd column from 2nd, and so on till the last column of A has been taken into consideration. Thus C will be an $n \times (n-1)$ matrix. Likewise, obtain a new matrix R of order $(n-1) \times n$ by subtracting its successive row's from the preceding ones as is done for columns to obtain C as above.

***Step* 2.** Augment the matrix A as below :

$$\begin{array}{c|c} \boxed{\quad A \quad} & C \\ R & \\ \hline & \end{array}$$

***Step* 3.** Compute the magnitude of oddments corresponding to each row and each column of A.

The oddment corresponding to ith row of A is defined as the determinant $|C_i|$, where C_i is obtained from C by deleting its ith row.

Similarly, oddment corresponding to jth column of $A = |R_j|$, where R_j is obtained from R by deleting its jth column.

***Step* 4.** Write down the magnitude of oddments (ignoring the –ve sign, if any) against their respective rows and columns as shown is *step 2*.

***Step* 5.** Now check whether the sum of row oddments is equal to the sum of column oddments.

(i) If so, the oddments expressed as the fractions of the grand total will provide the optimal strategies.

(ii) If not, the method fails.

Q. 1. Briefly explain the matrix method and short-cut matrix method for solving a rectangular game.

2. Show that if the elements of a payoff matrix are all integers, then the value of the game is a rational number.

3. Show that the game with payoff matrix $\begin{pmatrix} a & 0 & 1 \\ 1 & a & 0 \\ 0 & 1 & a \end{pmatrix}$ has a unique solution. Determine the value of the game.

4. Show that the game whose payoff matrix is $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ where $a > b > c > 0$, has a unique solution. What is the value of the game ? What can be said about the solution if $a > b > c$ and $c < 0$?

The short-cut metrix method can be easily understood by the following numerical example.

Example 21. *Solve the following game by using short-cut matrix method.*

B

		I	II	III	Row min
A	I	7	1	7	1
	II	9	– 1	1	– 1
	III	5	7	6	(5) ←maximin value
Column max.		9	(7)	(7)	

minimax value

Solution.

***Step* 1.** The first step is to search for a saddle point. There is no saddle point in this problem, but the value of the game lies between 5 and 7 (*i.e.* between maximin and minimax values).

***Step* 2.** The next step is to observe if the given matrix can be reduced by dominance. We find that the given matrix cannot be reduced. So we now solve this matrix game by *method of matrices* as explained in the following steps.

***Step* 3.** Subtract each row from the row above it (*i.e.* subtract 2nd row from 1st and 3rd row from the 2nd) and write down the values below the matrix. Similarly, subtract each column from the column to its left (*i.e.* subtract 2nd column from the 1st and 3rd column from the 2nd) and write down the results to the right of the matrix. Thus, we get the following table.

		B				
		I	II	III		
	I	7	1	7	6	−6
A	II	9	−1	1	10	−2
	III	5	7	6	−2	1
		−2	2	6		
		4	−8	−5		

***Step* 4.** Now, compute the oddments for A_1, A_2, A_3 and B_1, B_2, B_3.

Oddment for $A_1 = \det \begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix} = 10 - 4 = 6,$

" " $A_2 = \det \begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix} = 12 - 6 = 6,$

" " $A_3 = \det \begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix} = -12 + 60 = 48$

Oddment for $B_1 = \det \begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix} = -10 + 48 = 38,$

" " $B_2 = \det \begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix} = 24 - 10 = 14,$

" " $B_3 = \det \begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix} = 16 - 8 = 8.$

***Step* 5.** Next, put down these oddments (as shown in the table below) neglecting their signs. Since both the sums of oddments are same (60 each), this is a solution to the game. If the sums are different, both players do not use all of their courses of actions in their strategies, and this method fails.

	I	II	III				
I	7	1	7	6	1	1/10	3/30
II	9	−1	1	6	1	1/10	3/30
III	5	7	6	48	8	8/10	24/30
	38	14	8	60			
	19	7	4				
	19/30	7/30	4/30				

Thus the optimum strategies are :

(i) *A* (3/30, 3/30, 24/30)

(ii) *B* (19/30, 7/30, 4/30)

(iii) Value of the game, $v = \frac{7 \times 1 + 9 \times 1 + 5 \times 8}{1 + 1 + 8} = 5\frac{3}{5}$.

Remark. The short-cut method of matrices can be applied only when the sum of *vertical oddments* is equal to the sum of *horizontal oddments*. In other words, if both the players use all their plays in their best strategies. The method breaks down when the players do not use all their courses of action in their best strategies. In such a case the method of linear programming may be used.

EXAMINATION PROBLEMS

Use matrix method to solve the games whose payoff matrices are given below (apply short-cut method wherever applicable):

1. (a) $\begin{bmatrix} 2 & -2 & 3 \\ -2 & 5 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

[Ans. (2/3, 1/3), (7/12, 0) , $v = 1/3$] **[Ans.** (1/2, 1/2), (0, 1, 0), $v = 1$] **[Ans.** (0, 1/2, 1/2), (0, 1/2, 1/2), $v = 0$]

3. Use matrix method to solve the game whose payoff matrix is

$$\begin{bmatrix} 2 & 3 & 11 \\ 7 & 5 & 2 \end{bmatrix}$$

[Ans. (9/11, 2/11), (0, 3/11, 8/11), $v = 49/11$]

4. Let the payoff matrix of a rectangular game be $\begin{bmatrix} \lambda & \lambda & \lambda \\ \lambda & 3 & 4 \\ \lambda & 5 & 1 \end{bmatrix}$. Show that for all values of λ, the value of the game is λ. For what value of λ does the row player (column player) have an infinite number of optimal strategies.

9.17. ALGEBRAIC METHOD FOR THE SOLUTION OF A GENERAL GAME

The algebraic method is a direct attempt to solve unknowns from the relationships (equations 9·21a) in *Sec.* **9·10** for player *A*, and similarly for player *B*. Although this method becomes quite lengthy when there are more strategies (courses of action) for players *A* and *B*. Such large games can be solved first by transforming the problem into a linear programming problem and then solving it by the *simplex method* on an electronic computer.

First, suppose that all inequalities given in (9·21a) hold as equations. Then solve these equations for unknowns. Sometimes equations are not consistent. In such cases, one or more of the inequalities must hold as strict inequalities (with '>' or '<' signs). Hence, there will be no alternative except to rely on trial-and-error method for solving such games. Following important theorems will be helpful in making the computations easier.

Theorem 9·7. *If for any j* ($j = 1, 2, 3, \dots, n$) $v_{1j}x_1 + v_{2j}x_2 + \dots + v_{mj}x_m > v$, *then* $y_j = 0$, *and similarly, if for any i* ($i = 1, 2, 3, \dots, m$) $v_{i1}y_1 + v_{i2}y_2 + \dots + v_{in}y_n < v$, *then* $x_i = 0$.

Alternative Statement : *Let v be the value of an* $m \times n$ *game. If for an optimum strategy* $\mathbf{x}^* \in S_m$, $E(\mathbf{x}^*, \mathbf{e_j}) > v$ *for some* $\mathbf{e_j} \in S_n$, *then every strategy* $\mathbf{y}^* \in S_n$ *has* $y_j^* = 0$,

Similarly, if every optimal strategy $\mathbf{y}^* \in S_n$, *then every optimal strategy* $\mathbf{x}^* \in S_m$ *has* $x_i^* = 0$.

Proof. We know that for any optimal strategy $\mathbf{x}^* \in S_m$, we always have

$$E(\mathbf{x}^*, \mathbf{e_j}) \geq v \text{ for all } \mathbf{e}_j \in S_n \qquad \dots(1)$$

We are given that $E(\mathbf{x}^*, \mathbf{e_j}) > v$ for some $\mathbf{e_j} \in S_n$. If possible let us suppose $y_j^* \neq 0$ (*i.e.* $y_j^* > 0$).

Then $\quad y_j^* E(\mathbf{x}^*, \mathbf{e_j}) > y_j^* v$. $\qquad \dots.(2)$

Now $\quad E(\mathbf{x}^*, \mathbf{y}^*) = \Sigma y_k^* E(\mathbf{x}^*, \mathbf{e_k}) \qquad = \sum_{k \neq j} y_k^* E(\mathbf{x}^*, \mathbf{e_k}) + y_j^* E(\mathbf{x}^*, \mathbf{e_j})$

or $\quad v > \sum_{k \neq j} y_k^* v + y_j^* v$ or $v > (\sum_{k \neq j} y_k^* + y_j^*)v$ or $v > \Sigma\, y_k^* v$ [from (1) and (2)]

or $\quad v > v$ (since $\Sigma y_k^* = 1$) which is a contradiction.

Hence our assumption is wrong; and therefore, we must have $y_j^* = 0$.

Similarly, the second part of the theorem can be proved.

Theorem 9.8. *Let v be the value of an* $m \times n$ *game. If* $\mathbf{y}^* \in S_n$ *is an optimal strategy for the column vector with* $y_j^* \neq 0$, *then every optimal strategy* $\mathbf{x}^* \in S_m$ *for the row player must satisfy* $E(\mathbf{x}^*, \mathbf{e_j}) = v$ for all $\mathbf{e_j} \in S_n$.

Similarly, if the optimal strategy $\mathbf{x}^* \in S_m$ *has* $x_i^* \neq 0$, *then every optimal strategy* $\mathbf{y}^* \in S_n$ *must satisfy* $E(\mathbf{e_i}, \mathbf{y}^*) = v$ *for all* $\mathbf{e}_i \in S_m$.

Proof. Left as an exercise to the students.

Theorem 9.9. *If the player A's optimal policy is a mixed strategy in which exactly* **r** *pure strategies have non-zero probabilities, then the player B's optimal strategy also uses* **r** *pure strategies.*

Proof. Left as an exercise to the students.

Q. 1. If $X = \{x_i\}$ is an optimal mixed strategy for *A* and $Y = \{y_j\}$ is optimal mixed strategy for *B*, in a rectangular game specified by an $m \times n$ matrix $\{v_{ij}\}$ and *v* is the value of the game, then prove that if $E(e_i, Y) < v$, then $x_i = 0$, where $\mathbf{e_i}$ is the *i* th unit vector in *m*-dimensions and $E(\mathbf{e_i}, Y)$ is the expected amount received by *A* when he uses the strategy $\mathbf{e_i}$ and *B* uses the strategy *Y*.

2. Let *E* be the expectation function of an $m \times n$ rectangular game whose value is *v* and let

$$Y^* = \| x_1^*, x_2^*, \dots, x_m^* \| \text{ and } Y^* = \| y_1^*, y_2^*, \dots, y_n^* \|$$

be any optimal strategies for P_1 and P_2, respectively. Then,

(a) for any *l* such that $E(l, Y^*) < v$, show that $x_1^* = 0$ (b) for any *j* such that $v < E(x^*, j)$ show that $y_j^* = 0$

[Delhi B.Sc. (Math.) 93]

9·17-1. Illustrative Examples

Example 22. *Find the value and optimal strategies for two players of the rectangular game whose payoff matrix is given by*

			B y_1 I	B y_2 II	B y_3 III
A	x_1	I	1	-1	-1
A	x_2	II	-1	-1	3
A	x_3	III	-1	2	-1

Solution. First, it is seen that this game does not have a saddle point. Also, this game cannot be reduced to (2 × 2) by the property of dominance. Hence, this game can be solved by the algebraic method.

Step 1. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) denote the optimal mixed strategies for players A and B, respectively, and v be the vlaue of the game to the player A.

Now, for the player A, following relationships (*as obtained in Sec. 9·10*) can be established :

$$1x_1 + (-1)\,x_2 + (-1)\,x_3 \geq v$$
$$-1\,x_1 + (-1)\,x_2 + \quad 2\,x_3 \geq v$$
$$-1\,x_1 + \quad 3x_2 + (-1)\,x_3 \geq v.$$

Similarly, for the player B,

$$1.y_1 + (-1)\,y_2 + (-1)\,y_3 \leq v$$
$$-1.y_1 + (-1)\,y_2 + \quad 3\,y_3 \leq v$$
$$-1.y_1 + \quad 2y_2 + (-1)\,y_3 \leq v.$$

Additional relationship required to ensure that x_1, x_2, x_3 and y_1, y_2, y_3 are probabilities, are :

$$x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0 \text{ and } y_1 + y_2 + y_3 = 1,\ y_1, y_2, y_3 \geq 0.$$

Now the values of seven unknowns x_1, x_2, x_3 ; y_1, y_2, y_3 and v satisfying above relationships are to be evaluated.

Step 2. Suppose all inequalities hold as equations, then

$$x_1 - x_2 - x_3 = v \quad \text{...(i)} \qquad y_1 - y_2 - y_3 = v \quad \text{...(v)}$$
$$-x_1 - x_2 + 2x_3 = v \quad \text{...(ii)} \qquad -y_1 - y_2 + 3y_3 = v \quad \text{...(vi)}$$
$$-x_1 + 3x_2 - x_3 = v \quad \text{...(iii)} \qquad -y_1 + 2y_2 - y_3 = v \quad \text{...(vii)}$$
$$x_1 + x_2 + x_3 = 1 \quad \text{...(iv)} \qquad y_1 + y_2 + y_3 = 1. \quad \text{...(viii)}$$

Now with the help of the equation (iv), equations (i), (ii) and (iii) give us

$$x_1 = \frac{v+1}{2},\ x_2 = \frac{v+1}{4},\ x_3 = \frac{v+1}{3}$$

and substituting these values of x_1, x_2, x_3 in equation (iv), we get

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $v + 1 = 12/13$ or $v = -1/13$.

Hence $x_1 = 6/13, x_2 = 3/13, x_3 = 4/13$.

Again with the help of equation (viii), equations (v), (vi) and (vii) give us

$$y_1 = \frac{v+1}{2},\ y_2 = \frac{v+1}{3},\ y_3 = \frac{v+1}{4}$$

and substituting these values of y_1, y_2, y_3 in equation (viii),

$$\frac{v+1}{2} + \frac{v+1}{4} + \frac{v+1}{3} = 1.$$

Therefore, $v + 1 = 12/13$ or $v = -1/13$, which was expected also by minimax theorem.

Thus, $y_1 = 6/13, y_2 = 4/13$ and $y_3 = 3/13$.

Hence, the solution of the game is :

(*i*) Optimal mixed strategy for the player A is (6/13 , 3/13 , 4/13)

(*ii*) Optimal mixed strategy for the player B is (6/13, 4/13, 3/13)

(*iii*) The value of the game to the player A is = − 1/13.

Example 23. *In the following* 3 × 3 *game, find optimal strategies and the value of the game.* **[Delhi (OR.) 92]**

Table 9·39

		B I	B II	B III
A	I	3	-2	4
A	II	-1	4	2
A	III	2	2	6

Solution. It can be observed that this game does not have a saddle point. The size of the matrix of this game can be further reduced by using the dominance property as follows :

Step 1. From the player B's point of view, every element of IIIrd column is greater than the corresponding element in the Ist column. Hence the column III is dominated by the Ist column. So the size of the game can be reduced by deleting III column.

Step 2. Further, none of the rows or columns dominates the other. But, the average of I and II rows in the reduced matrix (a), *i.e.*

$$\frac{3+(-1)}{2}=1, \frac{-2+4}{2}=1,$$

is less than the corresponding elements of the III row. Hence the III row dominates the average of I and II rows. Thus, delete either I or II row. If we delete the I row, the 2 × 2 reduced game (b) is obtained.

(a)

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & \text{I} & \text{II} \\ \hline & \text{I} & 3 & -2 \\ \hline A & \text{II} & -1 & 4 \\ \hline & \text{III} & 2 & 2 \\ \hline \end{array}$$

(b)

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & \text{I} & \text{II} \\ \hline A & \text{II} & -1 & 4 \\ \hline & \text{III} & 2^* & 2 \\ \hline \end{array}$$

(c)

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & \text{I} & \text{II} \\ \hline A & \text{I} & 3 & -2 \\ \hline & \text{III} & 2 & 2^* \\ \hline \end{array}$$

Step 3. Now, the cell (III, I) is the saddle point. For element '2' marked '*' is minimum in its row and maximum in its column.

Thus, the solution of the game is given by

(i) Optimal strategy for the player A is pure strategy (0, 0, 1),

(ii) Optimal strategy for the player B is also pure strategy (1, 0, 0),

(iii) The value of the game to the player A is 2.

Remark. If IInd row is deleted, (instead of Ist in the reduced 3 × 2 matrix) the saddle point is obtained in the cell (III, II), as shown in matrix (c) above. Consequently, the optimal strategy for the player A becomes (0, 0, 1) and for the player B becomes (0, 1, 0). The value of the game will not change.

Example 24. *Solve the game with the following payoff matrix by algebraic method.*

Table 9.40

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{B} \\ & & \text{I} & \text{II} \\ \hline A & \text{I} & 3 & 2 \\ \hline & \text{II} & 4 & 1 \\ \hline \end{array}$$

Solution. The game has a saddle point (2) which immediately gives the required solution, but it will be solved by algebraic method to make the procedure clear.

Let (x_1, x_2) and (y_1, y_2) be the optimal strategies for player A and B, respectively.

Therefore, relationships existing for optimality are:

$$3x_1+4x_2 \geq v \quad \text{...(i)} \qquad 3y_1+2y_2 \leq v \quad \text{...(iv)}$$

$$2x_1+1x_2 \leq v \quad \text{...(ii)} \qquad 4y_1+y_2 \leq v \quad \text{...(v)}$$

$$x_1+x_2=1 \quad \text{...(iii)} \qquad y_1+y_2=1 \quad \text{...(vi)}$$

and

$$x_1, x_2, y_1, y_2 \geq 0. \quad \text{...(vii)}$$

First, suppose all inequalities hold as equations, *i.e.*

$$3x_1+4x_2=v, \qquad 2x_1+x_2=v, \qquad x_1+x_2=1$$

$$3y_1+2y_2=v, \qquad 4y_1+y_2=v, \qquad y_1+y_2=1.$$

Eliminate x_2 and y_2 from equations (i), (ii), (iv) and (v) with the help of equations (iii) and (vi), to obtain

$$-x_1+4=v, \qquad x_1+1=v, \qquad y_1+2=v, \qquad 3y_1+1=v,$$

or

$$x_1=3/2, y_1=1/2.$$

But x_1 can never be greater than unity (by the definition of probability). Thus, equations are not consistent. Hence one or more of the inequalities must hold as strict inequalities.

Now, use trial and error method together with the *Theorems* 9.4 & 1.5. Let two of the equations be strict inequalities:

$$-x_1+4>v \qquad \text{(which implies } y_1=0)$$

$$x_1+1=v$$

$$y_1+2=v$$

$$3y_1+1<v \qquad \text{(which implies } x_2=0).$$

Since $y_1=0$, $v=2$. Now, put $v=2$ in $x_1+1=v$ to get the value $x_1=1$.

Hence the solution is given by :

(i) Optimal strategy of the player A is $(x_1, x_2) = (1, 0)$

(ii) Optimal strategy of the player B is $(y_1, y_2) = (0, 1)$,

(iii) The value of the game to the player A is 2.

Remark. In view of the Theorem 1.9 (**Sec. 9·17**), the total number of equations that remain to solve (excluding the strict inequalities) at each trial should always be even.

EXAMINATION PROBLEMS

Solve the following games by algebraic method.

1. $\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} -1 & -2 & 8 \\ 7 & 5 & -1 \\ 6 & 0 & 12 \end{bmatrix}$ **[JNTU 97]** **3.** $\begin{bmatrix} -2 & -4 \\ -1 & 3 \\ 1 & 1 \end{bmatrix}$ **[JNTU (B. Tech.) 2003]**

[Ans. (2/9, 5/9, 2/9), $v = 1/9$].

4. $\begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix}$ **[Meerut 90]** **5.** $\begin{bmatrix} 3 & -1 & 1 & 2 \\ -2 & 3 & 2 & 3 \\ 2 & -2 & -1 & 1 \end{bmatrix}$

[Ans. (7/23, 6/23, 10/23) ; (17/46, 20/46, 9/46), $v = 15/23$]

6. $A = \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix}$ **7.** $\begin{bmatrix} 2 & 3 & -1 \\ 4 & -1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

[Ans. (4/11, 7/11), (0, 7/11, 4/11), $v = -5/11$] **[Ans.** (8/30, 5/30, 17/30), (13/30, 11/30, 6/30), $v = 53/30$]

8. Consider the game with the following payoff matrix. Verify that the optimal strategy for either player is to mix his 3 pure strategies equally, what is the value of this game.

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

[Ans. (1/3, 1/3, 1/3) for both players, $v = 0$]

9. Solve the following zero-sum game for two persons. Obtain the best strategies for both players and the value of the game :

$$\begin{array}{cc} & \textbf{Player } B \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \textbf{Player } A \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 2 \\ -1 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix} \end{array}$$

[Ans. A (0, 1/4, 3/4, 0), B (1/4, 3/4, 0), $v = -1/4$]

10. Use algebraic method to solve the games whose payoff matrices are :

(a) $\begin{bmatrix} 1 & -1 & 0 \\ -6 & 3 & -2 \\ 8 & -5 & 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 4 & -2 \\ -3 & 0 & 1 \\ -1 & -4 & 2 \end{bmatrix}$ **(c)** $\begin{bmatrix} 5 & 2 & -3 \\ 4 & 2 & 7 \\ -4 & 5 & -2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}$

[Ans. (a) (0, 7/12, 5/12), (0, 1/3, 2/3), $v = -1/3$. (b) (21/52, 12/52, 19/52), (2/13, 3/13, 8/13), $v = 2/13$.
(c) (39/118, 54/118, 25/118), (30/118, 3/118, 85/118). (d) (4/9, 11/45 , 14/45), (14/45, 11/45, 4/9) $v = -29/45$]

11. Solve the game whose payoff matrix is given by

$$\begin{bmatrix} 1 & 7 & 2 \\ 0 & 2 & 7 \\ 5 & 1 & 6 \end{bmatrix}$$

[JNTU (B. Tech.) 98]

[Ans. (2/5, 0, 3/5), (3/5, 2/5, 0), $v = 17/5$]

12. Use concept of dominance to reduced the size of the matrix of the given problem to 2 × 3 matrix and solve the game. The payoff matrix of the game is

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

[Ans. (1/6, 5/6, 0), (0, 1/3, 2/3), $v = 14/3$]

9.18. AN ITERATIVE METHOD FOR APPROXIMATE SOLUTION

The algebraic method is generally adopted to solve the game for which the graphical method cannot be applied, but the games with large payoff matrices are extremely tiresome to solve by algebraic method. For such large games the iterative method is also very powerful to hand as well as machine computations. By this method, the approximate value of the game can be evaluated upto any desired degree of accuracy. Optimal strategies can also be determined, but not so satisfactorily.

While adopting this method, it is assumed that each player acts under the assumption that past is the best guide to the future and will play in such a manner so as to maximize his expected gain (or to minimize his expected loss).

This method can best be explained by the following example.

		B		
		I	II	III
	I	1	−1	−1
A	II	−1	−1	3
	III	−1	2	−1

Example 25. *Find the value and optimal strategies for two players of the rectangular game whose payoff matrix (for the player A) is given below :* **[Meerut 2002; Delhi (OR.) 93; Jodhpur M.Sc. (Math) 92]**

Solution. In this method, the player A arbitrarily selects any row and places it under the matrix. Here, select I row arbitrarily. The player B examines this row and chooses a column corresponding to the *smallest* number in the row. This is column III. Column III is then placed to the right of the matrix. The player A examines this column and chooses a row corresponding to the *largest* number in this column. This is row II. Row II is then added to the first row last chosen and the sum of the two rows is placed beneath the row last chosen. The player B chooses a column corresponding to the smallest number in the new row and adds this column to the column last chosen. In the case of a tie (equality of numbers which prevents either player from being victorious) that row or column must be chosen by a coin flipping process. But, in this example, in case of a tie, the player will select the row or column different from his last choice. This procedure may be continued in the like manner. Ten iterations are shown with smallest elements in each succeeding row and largest elements in each succeeding column encircled. Approximate strategies after *ten* iterations are found by dividing the number of encircled elements in each row or column by total number of iterations. Thus the player A's approximate strategy is $(5/10, 2/10, 3/10)$, and the player B's approximate strategy is $(6/10, 2/10, 2/10)$, (*Table 9·41*)

Table 9·41

		B													
		I	II	III											
	I	1	−1	−1	$-\boxed{1}$	$-\boxed{2}$	−3	−2	−1	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{2}$	5/10
A	II	−1	−1	3	3	2	1	0	−1	−2	−3	−4	−5	−2	2/10
	III	−1	2	−1	−1	1	$\boxed{3}$	$\boxed{2}$	$\boxed{1}$	0	−1	−2	−3	−4	3/10
		1	−1*	$\textcircled{-1}$											
		0	−2	2											
		$\textcircled{-1}$	$\textcircled{-3}$	5											
		$\textcircled{-2}$	−1	4											
		$\textcircled{-3}$	1	3											
		$\textcircled{-4}$	3	2											
		$\textcircled{-3}$	2	1											
		$\textcircled{-2}$	1	0											
		$\textcircled{-1}$	0	−1*											
		0	−1	$\textcircled{-2}$											
		6/10	2/10	2/10											

The upper bound for the value of the game can be determined by dividing the largest element, 2, in the last column by the total number of iterations, 10. Likewise, the lower bound can be determined by dividing the smallest element, − 2, in the last row by the number of iterations, 10.

Thus, $-2/10 < v < 2/10$ or $-1/5 < v < 1/5$.

The approximate solution thus obtained is given by :

(*i*) Optimal strategy for player A is $(5/10, 2/10, 3/10)$

(*ii*) Optimal strategy for player B is $(6/10, 2/10, 2/10)$

(*iii*) The value of the game lies between $1/5$ and $-1/5$.

Other approximate solutions can also be obtained with more iterations.

Q. 1. Explain the iterative method of getting an approximate solution to a game problem.

2. Discuss whether the sequence of approximate optimal strategies for the row (column) player converges.

3. Show that for a 2×2 game with a saddle point, the iterative method leads to the true value of the game.

EXAMINATION PROBLEMS

1. Obtain an approximate solution by *Iterative method.* If possible, find the exact solution also.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix} \end{array}.$$

[Ans. (1/4, 1/4, 1/2), (1/3, 1/3, 1/3), $v = 1$]

2. Solve the following game approximately :

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 3 & -1 & 0 \\ 5 & 4 & 2 & -2 \\ 1 & 3 & 8 & 2 \end{bmatrix} \end{array}$$

[Ans. (0, 1/10, 9/10), (4/10, 0, 0, 6/10), $14/10 \le v \le 16/10$]

3. Obtain the approximate solution by iterative method of the games given below. If possible, find the exact solution also :

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 3 \\ 6 & 2 & 1 \end{bmatrix}$

[Ans. (a) (4/10, 3/10, 3/10), (5/10, 2/10, 3/10), $8/10 < v < 12/10$.
(b) (5/10, 0, 5/10), (2/10, 7/10, 1/10), $25/10 < v < 29/10$]

4. Find to an accuracy of two places of decimals, the value of the games whose payoff matrices are :

(a) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 5 & 0 \end{bmatrix}$

[Ans. (a) (1/3, 1/3, 1/3), (1/3, 1/3, 1/3), $v = 0{\cdot}67$. (b) (5/8, 1/8, 2/8), (3/8, 3/8, 2/8), $13/8 < v < 15/8$]

5. Show by the iterative method, that the game whose payoff matrix is $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ has a value zero.

9.19. SUMMARY OF METHODS FOR RECTANGULAR GAMES

In order to solve the two-person zero-sum games, we must proceed in the systematic order as follows :

***Step* 1.** First of all search for a saddle point. If it is found, the problem is readily solved.

***Step* 2.** If no saddle point is found, then use the concept of dominance to reduce the size of the matrix game. If dominance is found, delete the dominated row(s) and/or column(s). Each matrix thus obtained must be further checked for dominance.

***Step* 3.** If the size of the reduced matrix become (2×2) with no saddle point, it can be solved by *arithmetic* and *algebraic* methods described in ***Sections* 9·13–1 & 9·17.**

***Step* 4.** If the size of the reduced matrix becomes $(2 \times n)$ or $(m \times 2)$, use graphical method to reduce it to (2×2) matrix and then solve it by arithmetic or algebraic method.

If graphical method is not to be used, the game can still be solved by algebraic method and method of subgames. All these methods are described in ***Sections* 9·15-4 & 9·17.**

***Step* 5.** If the reduced size of the matrix becomes (3×3) or higher, then *algebraic method, method of matrices, simplex method of linear programming,* or *the iterative method of approximate solution* can be used for solving it. These methods are discussed in ***Sections* 9·17, 9·16, 9·12 & 9·18.**

Q. 1. "Game theory provides a systematic quantitative approach for analysing competitive situations in which the competitions make use of logical processes and techniques in order to determine an optimal strategy for winning." Comment.

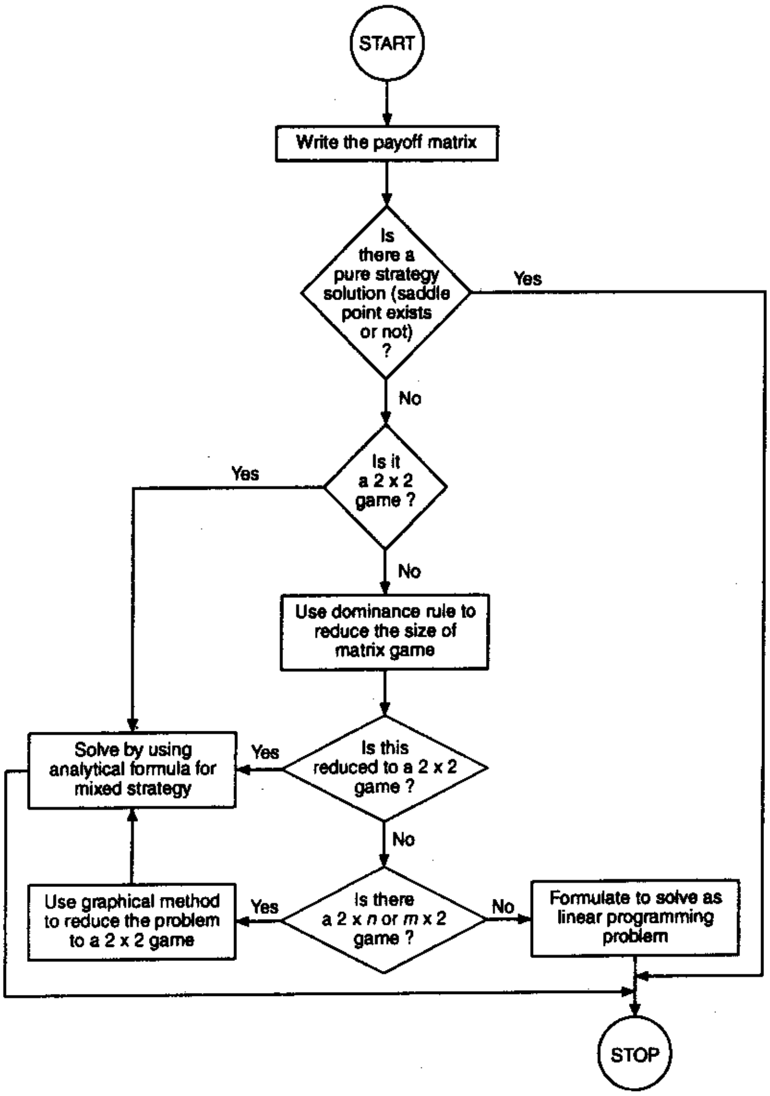
2. Summarize the systematic methods for solving the rectangular games.

9·20. BUSINESS APPLICATIONS : BIDDING PROBLEMS

Business problems often involve bidding for contracts, *e.g.*, getting the opportunity for service or bidding for the rights to get some privileges such as *contracts, land, property, licences* and *concessions*, etc. The types of bidding problems are : *(i)* Open or Auction bids, *(ii)* Closed bids.

Some simple bidding problems can be easily done by the techniques of game theory. Following examples will illustrate the procedure.

FLOW CHART FOR SYSTEMATIC APPLICATION



Example 26. (Open or Auction Bids). *Two items of worth Rs. 75 and Rs. 125 are to be auctioned at a public sale. There are only two bidders A and B. Bidder A has Rs. 100 available and bidder B has Rs. 130 available. What should be their strategies if each bidder is interested in maximizing his own return.*

[Meerut (Maths.) 98 BP]

Solution. Let the successive increase of bids be Rs. λ. At any bid, each bidder has an option to increase the bid or to leave the opponents bid stand. Further, let B has bid, Rs. x on the Ist item (of value Rs. 75). Then A will think as follows :

If bidder A permits B to win the Ist item for Rs. x, then B will have only Rs. $(130 - x)$ for bidding on the IInd item. Obviously, B cannot bid for the IInd item more than Rs. $(130 - x)$. So, A will surely win the IInd item in Rs. $(130 - x + \lambda)$. hence the A's gain allowing B to win the Ist item for Rs. x, will be

$$\text{Rs. } [125 - (130 - x + \lambda)] \quad \text{or} \quad \text{Rs. } [x - \lambda - 5].$$

Alternatively, if A bids for Rs. $(x + \lambda)$ for the Ist item and B permits him to win at this bid, the A's gain will be Rs. $[75 - (x + \lambda)]$ or Rs. $[75 - x - \lambda]$.

Since A wishes to maximize his return, he should bid $x + \lambda$ for the Ist item, subject to the condition

$$75 - x - \lambda \geq x - \lambda - 5 \text{ or } x \leq 40.$$

Therefore, *A* should bid for the Ist item till $x \leq 40$. In case $x > 40$, he should allow *B* to win the Ist item for that bid.

Likewise, *B*'s gains in two alternatives are : Rs. $[125 - (100 - y) - \lambda]$ and Rs. $[75 - (y + \lambda)]$ where *y* denotes the *A*'s bid for the first item.

So *B* should bid Rs. $(y + \lambda)$ for the Ist item subject to the condition :

$$75 - (y + \lambda) \geq 125 - (100 - y) - \lambda \text{ or } y \leq 25.$$

Clearly, *A* will then purchase the Ist item in Rs. 25, because he can increase his bid without any loss upto Rs. 40, and *B* will purchase the IInd item in Rs $(100 - 25) = 75$, because *A* (after winning the Ist in Rs. 25) cannot increase his bid for the IInd item more than Rs. 75. Therefore, *B* will purchase the second item in Rs. 75. So the gain to *A* is Rs. $(75 - 25) =$ Rs. 50 and to *B* is Rs. $(125 - 75) =$ Rs. 50.

Note. In this problem each bidder has an amount less than the total values of the two items. It is also assumed here that *A* knows the amount available with *B*.

Example 27. (Closed Bids). *Two objects of worth Rs. 80 and Rs. 100 are to be bid simultaneously by two bidders A and B. Both have intention of devoting a total sum of Rs. 100 to the two bids. If each uses a minimax criterion, find the resulting bids.*

Solution. In this problem, bids are closed because they are to be made simultaneously.

Suppose Rs. a_1 and Rs. a_2 are the *A*'s optimum bids for the Ist and IInd object respectively. *A*'s best bids are those which give the same amount of gain to *A* on both items. If *p* denotes the total profit associated with a successful bid, then

$$2p = (80 - a_1) + (100 - a_2) \text{ or } 2p = 180 - (a_1 + a_2).$$

Since both have decided to spend only Rs. 110 for both the bids, $(a_1 + a_2) = 110$. Therefore,

$$2p = 180 - 110 \text{ or } p = \text{Rs. } 35.$$

Hence $a_1 = 80 - p = 80 - 35 = \text{Rs. } 45$ and $a_2 = 100 - p = 100 - 35 = \text{Rs. } 65.$

Thus the optimum bids for *A* are Rs. 45 and Rs. 65 for the Ist and IInd items, respectively.

Proceeding likewise, *B*'s optimum bids can be determined. The optimum bids for *B* will be the same as that of *A*'s optimum bids.

Q. 1. What is game theory ? Discuss its importance to business decisions.
2. What is game theory ? Include in your answer various approaches in solving strategies and game values.
3. Describe the role of 'theory of games' for scientific decision making.
4. "A game refers to a situation of business conflict". Discuss.
5. Describe some of the applications of game theory. What are its limitations ?

9·21. LIMITATIONS OF GAME THEORY

Game theory which was initially received in literature with great enthusiasm as holding promise, has been found to have a lot of limitations. The major limitations are summarised below :

1. The assumption that the players have the knowledge about their own payoffs and payoffs of others is rather unrealistic. He can only make a guess of his own and his rivals' strategies.
2. As the number of players increase in the game, the analysis of the gaming strategies become increaseingly complex and difficult. In practice, there are many firms in an oligopoly situation and game theory cannot be very helpful in such situations.
3. The assumptions of maximin and minimax show that the players are risk-averse and have complete knowledge of the strategies. These do not seem practical.
4. Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion, Then the mixed strategies are not very useful.

However, inspite of its limiations, game theory provides insight into the operations of oligopoly markets.

Q. Briefly explain the major limitations of game theory. [JNTU (B. Tech.) 2003; Delhi (M.Com.) 90]

Q. 1. Define a saddle point. State and prove the necessary and sufficient condition for a function $f(x, y)$ to possess a saddle point.

2. Let f be a function of two variables such then $f(x, y)$ is a real number whenever $x \in A$ and $y \in B$. Suppose that $\underset{x \in A}{\text{Max}}\ \underset{y \in B}{\text{Min}}\ f(x, y)$ and $\underset{y \in B}{\text{Min}}\ \underset{x \in A}{\text{Max.}}\ f(x, y)$ both exist, then

prove that $\underset{x \in A}{\text{Max}}\ \underset{y \in B}{\text{Min}}\ f(x, y) \le \underset{y \in B}{\text{Min}}\ \underset{x \in A}{\text{Max}}\ f(x, y)$,

What conclusion is drawn when

$\underset{x \in A}{\text{Max}}\ \underset{y \in B}{\text{Min}}\ f(x, y) \le \underset{y \in B}{\text{Min}}\ \underset{x \in A}{\text{Max}}\ f(x, y)$, **[Delhi (OR.) 92]**

3. Discuss the saddle value problem and properties of optimal strategies in matrix games. **[Delhi (OR.) 95]**

SELF EXAMINATION QUESTIONS

1. What is a rectangular game ? Define 'pure strategy' and 'mixed strategy' in a game.

2. Explain the following terms : (i) pure strategy, (ii) Mixed strategy, and (iii) Optimal strategies.

3. Show how to solve a 2×2 two-person zero-sum game without any saddle point. Derive the expression for optimal strategies and the value of game.

4. Define mixed strategy and the value of a game in the theory of games. If a constant is added to each element of the payoff matrix, determine whether the set of optimal strategies of each player is the same or not. How does the value of the game then change ?

5. Explain what is meant by mixed extension of a rectangular game.

6. Define the expectation function in $m \times n$ rectangular game between two players. Enumerate and explain (without proof) the theorem for rectangular games in terms of the saddle point of the expectation function.

7. Let v be the value of a rectangular game with payoff matrix $B = (p_{ij})$.

(i) Show that $\min p_{ij} \le v \le \max p_{ij}$ and $\underset{i}{\max}\ \underset{j}{\min}\ p_{ij} \le v \le \underset{j}{\min}\ \underset{i}{\max}\ p_{ij}$.

(ii) For any unilateral deviation from optimal strategies, show that the expected yield will be unfavourable to the player who deviates from his optimal strategy.

(iii) If one of the players adheres to his optimal mixed strategy, show that the value of the game remains unaltered if the opponent uses the supporting strategies only, either or in a mixture.

8. Given the 2×2 payoff matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose player A adopts the strategy (x, y), while B adopts the strategy (u, v) where x, y, u, v are all ≥ 0, s.t. $x + y = u + v = 1$.

(i) Express A's expected gain z in terms of x, y, u, v and a, b, c, d.

(ii) What is the effect on adding the same constant k to each element of the payoff matrix ?

(iii) What is the effect on z of multiplying each element of payoff matrix by the same constant k?

(iv) How are the optimal strategies affected by these operations on payoff matrix.

9. (a) Show that the 2×2 game $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-strictly determined if,

(i) $a < b, a < c, d < b$ and $d < c$ or (ii) $a > b, a > c, d > b, d > c$.

(b) If G is a $p \times q$ matrix game with optimal strategies p and q and the value v, what can you say about the optimal strategies and the value of the matrix game $mG + nE$? Prove your result.

Here $m > 0$, n is a real number and E is a $p \times q$ matrix with all its elements unity.

10. Prove that the following two theorems are equivalent.

(i) In an $m \times n$ rectangular game, a saddle point (x_0, y_0), $x_0 \in S_m$, $y_0 \in S_n$ always exists such that

$$E(x, y_0) \le E(x_0, y_0) \le E(x_0, y)$$

where S_r is the set of r-tuples of real non-negative numbers, subject to the condition that sum of the elements of each r-tuples is unity.

(ii) $E(\theta_i, y_0) \le E(x_0, y_0) \le E(x_0, \phi_j)$, where $1 \le i \le m$, $1 \le j \le n$, and $\theta_i \in S_m$, $\phi_j \in S_n$ such that one element of the m-tuple θ_i is unity and all others are zero, and similarly for ϕ_j.

11. If $X = \{x_i\}$ is optimal mixed strategy for A and $Y = \{y_j\}$ is optimal mixed strategy for B in a rectangular game specified by an $m \times n$ matrix (a_{ij}), and V is the value of the game, then prove that if $E[e_i, Y] < V$, then $x_i = 0$, where e_i is the ith unit vector in n-dimensions, and $E[e_i, Y]$ is the expected amount received by A when he uses the strategy e_i and B uses the strategy Y.

12. Let $f(x, y)$ be real valued function defined for $x \in A$, $y \in B$, A and B being two sets.

Show that $\underset{x \in A}{\max}\ \underset{y \in B}{\min}\ f(x, y) \le \underset{y \in B}{\min}\ \underset{x \in A}{\max}\ f(x, y)$, provided that both exists. Further show that if (x_0, y_0) is a saddle point of $f(x, y)$, then

$$\max_{x \in A} \min_{y \in B} f(x, y) = (x_0, y_0) = \min_{y \in B} \max_{x \in A} f(x, y).$$

13. If all the elements of payoff matrix of a game are non-negative and every column of this matrix has at least one positive element, show that the value of the corresponding game is positive.

14. (a) Express a linear programming problem as a matrix game.

(b) What is a symmetric game ? Show that the value of a symmetric game is zero and that both players have identical optimal strategies. **[Delhi (OR.) 95, 93]**

15. "Game theory provides a systematic quantitative approach for analysing competitive situations in which the competitors make use of logical processes and techniques in order to determine an optimal strategy for winning" comment. **[Delhi MBA (Pt) 95]**

16. Give the comprehencive explanation of the term Game theory?

SELF EXAMINATION PROBLEMS

1. The payoff matrix for a 2-person, zero-sum game is given :

$$\begin{array}{cc} & \text{Player } A \\ & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{Player } B \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \end{array}$$

Stating the criterion you adopt, find the optimal strategies for two players when a, b and c are all of the same sign. What simplifications are effected for the case, when a, b and c are not of the same sign ? What is the value of the game in either case ?

2. Given the following payoff table :

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & -1 & 2 \\ 4 & 4 & -3 \\ 0 & 3 & -4 \end{bmatrix} \end{array}$$

(i) Determine the value of th game, if possible.
(ii) What is the minimax criterion for stable games ?
(iii) What is minimax criterion for unstable game ?
(iv) Formulate the problem as a linear programming problem (No derivation of the problem is required. Just state the objective function, constraints etc.)

3. In a well-known children's game, each player says 'stone' or 'scissors' or 'papers'. If one says 'stone' and the other 'scissors' then the former wins a rupee. Similarly 'scissors' beats 'paper' and 'paper' beats 'stone', *i.e.*, the player calling the former word wins a rupee. If the two players name the same item, then there is a tie, *i.e.*, there is no payoff. Write down the payoff matrix and the L.P. problem of either of the two players. Find the value of the game and hence write down the optimal strategies of both players. **[Delhi (OR.) 90]**

[Hint. Let A be the row player and B be the column player. Then the payoff matrix for player A is :

$$\begin{array}{cc} & \begin{array}{ccc} \text{Stone} & \text{Paper} & \text{Scissors} \end{array} \\ \begin{array}{c} \text{Stone} \\ \text{Paper} \\ \text{Scissors} \end{array} & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

4. The following matrix represents the payoff to P_1 in a rectangular game between two persons P_1 and P_2 :

$$\begin{array}{cc} & P_2 \\ P_1 & \begin{bmatrix} 8 & 15 & -4 & -2 \\ 19 & 15 & 17 & 16 \\ 0 & 20 & 15 & 5 \end{bmatrix} \end{array}$$

By the notion of dominance, reduce the game to 2×4 game and then solve it graphically.
[Ans. (0, 15/16, 1/16), (0, 11/16, 0, 5/16), $v = 245/16$.]

5. Consider the following military situation. Side I is interested in bombing two areas, each containing a vast factory complex of side II. However, side I has only one bomber squadron to devote to this, and the entire squadron must be sent to only one target area in order to be effective. Side II can defend effectively against such an attack if it can send all of its available fighters to meet the bomber squadron. However, the two areas are so far apart that these fighters can be available for defending only one of the areas at any particular time.

If they are not defended by the fighters, the military value of bombing target area II is considered to be 2·5 times that of bombing target area I. However, if they are defended. the bombers must turn back with heavy losses without reaching the target. The military value of this loss is considered to be equal in magnitude to the gain if target area I were to be bombed undefended.

Use game theory to formulate this problem, and to determine the optimal mixed strategy of the respective side according to the minimax criterion.

6. Two items of value Rs. 100 and Rs. 130 are to be auctioned at a public sale. Only two bidders are interested in these items. Bidder A has Rs 100 available and bidder B has Rs. 80 available. What should be their strategies if each bidder is interested in maximizing his own gain.
[Hint. Proceed exactly as solved example]

7. Country A has two Ammuntion stores, one of which is twice as valuable as the other. B is an attacker who can destroy an undefended store but he can attack any one of them. A knows that B is about to attack one of the stores but does not know what should he do ? Note that A can successfully defend only one store at a time. What should A do to maximize his return ?
[Ans. A : (1/3, 2/3), $v = -2/3$]

8. The labour contract between your management and the union will terminate in the near future. A new contract must be negotiated preferably before the old one expires. You are a member of a management group charged with selecting a strategy for them during the coming negotiations. After a consideration of past experience, the group agrees that feasible strategies for the company and union are :

$$\begin{array}{cc} & \text{Union's Strategies} \\ \text{Company's Strategies} & \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \text{I} & 20 & 15 & 12 & 35 \\ \text{II} & 25 & 14 & 8 & 10 \\ \text{III} & 40 & 2 & 19 & 5 \\ \text{IV} & 5 & 4 & 11 & 0 \end{array} \end{array}$$

I. All out attack, hard aggressive bargaining.

II. A reasonably logical approach. III. A legalistic strategy.

IV. An agreeable conciliatory approach.

The payoff table is given here. Find optimal strategy for the company. Determine the worth of your negotiations.

[**Ans.** Company (17/20, 0, 3/20, 0), Union (0, 7/20, 13/20, 0), $v = 261/20 = 13$ approx.]

9. Even through there are several manufactures of Scooters, two firms with brand name Janata and Praja, control their market in Western India. If both manufacturers make model changes of the same type for this market segment in the same year, their respective market shares remain constant. Likewise, if neither makes model changes, then also their market shares remain constant.

The payoff matrix in terms of increased/decreased percentage market share under different possible conditions is given below :

		Praja		
		No change	Minor change	Major change
	No change	0	-4	-10
Janata	Minor change	3	0	5
	Major change	8	1	0

(i) Find the value of the game.

(ii) What change should Janata consider if this information is available only to itself ?

[**Hint.** This game has no saddle point. Making use of dominance principle, since the first row is dominated by the third one, we delete the first row. Similarly, first reduced payoff matrix then becomes :

$$\begin{array}{c|cc} & \text{Minor} & \text{Major} \\ \text{Minor} & 0 & 5 \\ \text{Major} & 1 & 0 \end{array}$$

10. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisement in any one of its product against the other. The payoff matrix is given below :

$$\begin{array}{cc} & \text{Competitor} \\ \text{Company} & \begin{array}{c|ccc} & \text{I} & \text{II} & \text{III} \\ \text{I} & 6 & 7 & 15 \\ \text{II} & 20 & 12 & 10 \end{array} \end{array}$$

What is the best strategy for the company as well as the competitor ? What is the payoff obtained by the company and the competitor in the long run ? Use graphical method or linear programming method to obtain the solution.

11. Two players are each provided with an ace of diamonds and an ace of clubs. Player P_1 is also given the two of diamonds and player P_2, the two of clubs. In the first move, P_1 shows of his cards and P_2 ignorent of P_1's choice, shows one of his cards. P_1 wins if the suits match and P_2 wins if they do not. The amount of payoff is the numerical value of the card shown by the winner. If both the twos are shown, the payoff is zero.

[**Ans.** (1/2, 1/2), (1/2, 1/2), $v = 0$.]

12. An enterprising young statistician believes that he has developed a system for winning a popular Lasvegas game. His colleagues do not believe that this is possible, so they have made a large bet with him. They bet that, starting with three chips, he will not have five chips after three plays of the game. Each play of the game involves betting any desired number of available chips and then either winning or losing this number of chips. The statistitian believes that his system will give him a probability of 2/3 of winning a given play of the game. Assuming he is correct, determine his optimal policy regarding how many chips to bet (if any) at each of the three plays of the game. The decision at each play should take into account the results of earlier plays. The objective is to maximize the probability of winning his bet with his colleagues.

13. Consider a modified form of "matching biased coins" game problem. The matching player is paid Rs. 8.00 if the two coins turn both heads and player is paid Rs. 3 when the two coins do not match. Given the choice of being the matching or non-matching player, which one would you choose and what would be your strategy ?

[**Ans.** (4/15, 11/15), (4/15, 11/15), $v = -1/15$]

14. Two firms A and B are competing for an increased market share. To improve its market share, both the firms decide to employ the following promotional strategies :

(A_1, B_1) = No promotion, (A_2, B_2) = Moderate promotion, (A_3, B_3) = Much promotion. The payoff matrix, shown in the following table, describe the increase in market share for firm A and decrease in market share for firm B :

		Firm B		
		B_1	B_2	B_3
Firm A	A_1	5	20	-10
	A_2	10	6	2

Determine the optimal strategies for each firm and the value of the game. **[Delhi MBA. (RT). 95]**

15. Two computer manufacturers A and B are attempting to sell computer systems to two banks 1 and 2. Company A has 4 salesmen, company B has only 3 available. The computer companies must decide upon how many salesmen to assign to sell on each bank. Thus company A can assign 4 salesmen to assign to bank 1 and none to bank 2 or three to bank 1 and one to bank 2, etc.

Each bank will buy one computer system. The probability that a bank will buy from a particualr computer company is directly related to the number of salesmen calling from that company relative to total salesmen calling. Thus, if company *A* assignes three men to bank 1 and company *B* assigns two men the odds would be three out of five that bank 1 would purchase company *A*'s computer system. (if none calls from either company the odds are one-half for buying either computer.

Let the pay-off be the expected number of computer systems that company *A* sells. (Then 2 minus this pay-off is the expected number company *B* sells).

What strategy would company *A* use in allocating its salesmen ? What strategy should company *B* use ? What is the value of the game to company *A* ? What is the meaning of the value of the game in this problem ?

[Delhi (MBA) April 95]

[**Ans.** The payoff matrix for company *A* :

$$\text{Company } A \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \overset{\text{Company } B}{\begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \\ \end{array}} \left[\begin{array}{cccc} 1/2 & 0 & 0 & 0 \\ 1 & 1/2 & 1/3 & 1/4 \\ 1 & 2/3 & 2/4 & 2/5 \\ 1 & 3/5 & 3/5 & 3/6 \\ 1 & 4/5 & 4/6 & 4/7 \end{array}\right]$$

Optimum strategy for *A* is A_5 and for *B* is B_4.

Value of the game = 4/7, *i.e.*, prob. of success for *A* is 4/7 or 51% supply.

16. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following pay-off matrix describes the increase in market share of Firm *A* and the decrease in market share for Firm *B*. Determine the optimum strategies for each firm.

		Firm *B*		
		No promotion	Moderate promotion	Much promotion
Firm *A*	No promotion	5	0	– 10
	Moderate promotion	10	6	2
	Much promotion	20	15	10

(i) Which firm would be winner, in terms of market share ?

(ii) Would the solution strategies necessarily maximize profits for either of the firms ?

(iii) What might the two firms do to maximize their profits ? **[Delhi (M.B.A.) Nov. 97]**

[**Ans.** Optimum strategy for both *A* and *B* is much promotive ? Value of game = 10].

17. (a) Solve the following 2-person, zero-sum game :

$$\textbf{Player } A \quad \begin{array}{c|ccc} & \multicolumn{3}{c}{\textbf{Player } B} \\ & B_1 & B_2 & B_3 \\ \hline A_1 & 10 & 5 & 7 \\ A_2 & 6 & 7 & 5 \\ A_3 & 7 & 6 & 7 \end{array}$$

[Delhi (M. Com.) 98]

(b) The pay-off matrix for a two person zero-sum game is given below. Find the best strategy for each player and the value of the game.

$$\textbf{Player } A \quad \begin{array}{c|ccccc} & \multicolumn{5}{c}{\textbf{Player } B} \\ & B_1 & B_2 & B_3 & B_4 & B_5 \\ \hline A_1 & -2 & 0 & 0 & 5 & 3 \\ A_2 & 3 & 3 & 1 & 2 & 2 \\ A_3 & -4 & -3 & 0 & -2 & 6 \\ A_4 & 5 & 3 & -4 & 2 & -6 \end{array}$$

18. Two leading firms (firm *A* and firm *B*), for year's have been selling suitings, which is but a small part of both firm's total sales. The Marketing Director of firm *A* raised the question. "What should his firm's strategies be in terms of advertising for the product in question ?" The systems group of the firm *A* developed the following data for varying degrees of advertising :

(i) No advertising, medium advertising and large advertising for both firms will result in equal market share.

(ii) Firm *A* with no advertising : 40 per cent of the market with medium advertising by firm *B* and 28 per cent of the market with large advertising by firm *B*.

(iii) Firm *A* using medium advertising : 70 per cent of the market with no advertising by the firm *B* and 45 per cent of the market with large advertising by firm *B*.

(iv) Firm *A* using large advertising : 75 per cent of the market with no advertising by firm *B*.

Based upon the above information, answer the marketing director's question. **[Sadar Patel (M.B.A.) 97]**

19. Solve the following game :

$$\text{Player } A \quad \begin{array}{c|cccccc|} & \multicolumn{6}{c}{\text{Player } B} \\ & I & II & III & IV & V & VI \\ 1 & 4 & 2 & 0 & 2 & 1 & 1 \\ 2 & 4 & 3 & 1 & 3 & 2 & 2 \\ 3 & 4 & 3 & 7 & -5 & 1 & 2 \\ 4 & 4 & 3 & 4 & -1 & 2 & 2 \\ 5 & 4 & 3 & 3 & -2 & 2 & 2 \end{array}$$

[Gujarat (M.B.A.) 97]

[**Hint.** Game has no saddle point. Use principle of dominance.
Ans. S_A = [0, 6/7, 1/7, 0, 0], S_B = [0, 0, 4/7, 3/7, 0, 0], v = 13/7]

20. Two leading firms *A* and *B* are planning to make fund allocation for advertising their product. The matrix given below shows the percentage of market shares of firm *A* and *B* for their various advertising policies :

		Firm *B*		
		No advertising	Medium advertising	Heavy advertising
Firm *A*	No advertising	60	50	40
	Medium advertising	70	70	50
	Heavy advertising	80	60	75

Find the optimum strategies for the two firms and the expected outcome when both the firms follow their optimum strategies. **[H.P. (M.B.A.) Jan. 99]**

[**Hint.** No saddle point. Use dominance. Solve the 2 × 2 game $\begin{pmatrix} 70 & 50 \\ 60 & 75 \end{pmatrix}$]

21. Obtain the optimum strategies for both the firms for the adjoining pay-off matrix :

$$\begin{array}{c} \\ \text{Firm } A \end{array} \begin{array}{c} \text{Firm } B \\ \begin{bmatrix} 12 & 10 & 8 \\ 14 & 14 & 10 \\ 16 & 12 & 15 \end{bmatrix} \end{array}$$

[A.I.M.A. (P.G. Dip. in Management), Dec. 96]

[**Hint.** No saddle point. Use dominance.
Ans. S_A = [0, 3/7, 4/7], S_B = [0, 5/7, 2/7], v = 90/7]

22. Assume that two firms are competing for market share for a particular product. Each firm is considering what promotional strategy to employ for the coming period. Assume that the following pay-off matrix describes the increase in market share for for Firm *A* and the decrease in market share for Firm *B*. Determine the optimum strategies for each firm.

$$\begin{array}{ll|ccc|} & & \multicolumn{3}{c}{\text{Firm } B} \\ & & \text{No promotion} & \text{Moderate promotion} & \text{Much promotion} \\ & \text{No promotion} & 5 & 0 & -10 \\ \text{Firm } A & \text{Moderate promotion} & 10 & 6 & 2 \\ & \text{Much promotion} & 20 & 15 & 10 \end{array}$$

(i) Which firm would be the winner, in terms of market share ?
(ii) Would the solution strategies necessarily maximize profits for either of the firms ? **[Delhi (M.B.A.) April 99]**

23. Two candidates, *X* and *Y*, are competing for the councillor's seat in a city municipal corporation, and *X* is attempting to increase his total votes at the expense of *Y*. The strategies available to each candidate involve personal contacts, newspaper insertions/speeches or television appearance/advertising. The increase in votes available to *X* given various combinations of strategies are given below. (Assume that this is a zero-sum game, *i.e.*, any gain of *X* is equal to the votes lost by *Y*). Determine the optimum strategies that should be adopted by *X* during his election compaign. How many votes should *X* gain by the following optimum strategy ?

$$\begin{array}{ll|ccc|} & & \multicolumn{3}{c}{Y} \\ & & \text{Personal contacts} & \text{Newspapers} & \text{Television} \\ & \text{Personal contacts} & 30{,}000 & 20{,}000 & 10{,}000 \\ X & \text{Newspapers} & 60{,}000 & 50{,}000 & 25{,}000 \\ & \text{Television} & 20{,}000 & 40{,}000 & 30{,}000 \end{array}$$

[Delhi (M.B.A.) Dec. 95]

24. Two separate firms (*A* and *B*) have for years been selling a competing product which forms a part of both firm's total sales. The marketing executive of firm *A* raised the question. "What should be the firm's strategies in terms of advertising for the product in question." The market research team of firm *A* developed the following data for varying degrees of advertising :

(i) No advertising, medium advertising, and large advertising for both firms will result in equal market shares.

(ii) Firm *A* with no advertising : 40% of the market with medium advertising by firm *B* and 28% of the market with large advertising by firm *B*.

(iii) Firm *A* using medium advertising : 70% of market with no advertising by firm *B* and 45% of the market with large advertising by firm *B*.

(iv) Firm *A* using large advertising : 75% of the market with no advertising by firm *B* and 47·5% of the market with medium advertising by firm *B*.

(a) Based upon the foregoing information, answer the marketing executive's question. **[Delhi (M.B.A.) Dec. 96; March 99]**

(b) What advertising policy should firm *A* pursue when consideration is given to the above factors : selling price, Rs. 4·00 per unit; variable cost of product, Rs. 2·50 per unit; annual volume of 30,000 units for firm *A* ; cost of annual medium advertising Rs. 5,000 and cost of annual large advertising Rs. 15,000 ? What contribution, before other fixed costs, is avaialble to the firm ? **[A.I.M.A. (P.G. dip. in Management), 97]**

25. A soft drink company calculated the market share of two products against its major competitor having three products and found out the impact of additional advertisment in any one of its products against the other.

$$\textbf{Competitor } B$$

$$\textbf{Company } A \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{ccc} B_1 & B_2 & B_3 \\ \left[\begin{array}{c} 6 \\ 20 \end{array}\right. & \begin{array}{c} 7 \\ 12 \end{array} & \left.\begin{array}{c} 15 \\ 10 \end{array}\right] \end{array}$$

What is the best strategy for the company as well as the competitor ? What is the pay-off obtained by the company and the competitor in the long run ? Use graphical method to obtain the solution. **[Delhi (M.B.A.) April 98]**
[**Ans.** Company *A* = [2/3, 1/3, 0], Competitor *B* = {7/12, 5/12], *v* = 1/3]

26. Firm *X* is fighting for its life against the determination of firm *Y* to drive it out of the industry. Firm *X* has the choice of increasing price, leaving it unchanged, or lowering it. Firm *Y* has the same three options. Firm *X*'s gross sales in the event of each of the pairs of choices are shown below :

Firm *Y*'s pricing strategies

		Increase price	Do not change	Reduce price
Firm *X*'s pricing strategies	Increase price	90	80	110
	Do not change	110	100	90
	Reduce price	120	70	80

Assuming firm *X* as the miximizing one, formulae the problem as a linear programming problem. **[Osmania (M.B.A.) Nov. 96]**

{**Ans.** Player *A* : (3/8, 13/24, 1/12), Player *B* : (7/24, 5/9, 11/22), *v* = 91/24}.

27. In zero-sum two person children's game of stone, paper and scissors, both players simultaneously call out stone, paper or scissors. The combination of paper and stone is a win of one unit for player calling paper (paper covers stone); stone and scissors is a win for stone (stone breaks scissors), and scissors and paper is a win for scissors (scissors cut paper). A call of the same item represents no pay-off. Write the pay-off matrix and the equivalent linear programign problem to the above game. Find the optimum strategy for both the players and the value of the game. **[A.I.M.A. (P.G. Dip. in Management), Dec. 95]**

28. In a town, there are only two discount stores *ABC* and *XYZ* . Both stores run annual pre-diwali sales during the first week of October. Sales are advertised through local newspapers with the aid of an advertising firm. *ABC* stores constructed following pay-off in units of Rs. 1,00,000. Find the optimum strategies for both stores and the value of the game ;

$$\textbf{Strategies of } XYZ$$

$$\textbf{Strategies of ABC} \quad \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & -2 \\ -1 & -2 & 3 \end{bmatrix} \end{array}$$

[Bombay (M.M.S.) 95]

29. Determine the saddle-point solution, the associated pure strategies and the value of the game whose pay-off matrix is given below :

	B_1	B_2	B_3	B_4
A_1	4	−4	−5	6
A_2	−3	−4	−9	−2
A_3	6	7	−8	−9
A_4	6	3	−9	5

[Meerut (MCA II) 2000]

30. Solve the game approximately :

$$\mathbf{A} \quad \begin{array}{c} \\ \mathbf{I} \\ \mathbf{II} \\ \mathbf{III} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{I} & \mathbf{II} & \mathbf{III} \end{array} \\ \begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix} \end{array}$$

[Meerut 2002]

31. Use dominance principle to simplify the rectangular game with the following payoff matrix and then solve graphically.

		Player B			
		I	II	III	IV
Player A	1	18	4	6	4
	2	6	2	13	7
	3	11	5	17	3
	4	7	6	12	2

[AIMS (MBA) 2002]

32. (a) Use the Dominance principle and solve the game :

		B		
		1	2	3
A	I	1	−3	−2
	II	0	−4	2
	III	−5	2	3

[VTU 2003]

(b) Solve the following game using graphical method.

		B		
		1	2	3
A	I	3	−1	0
	II			
	III	2	1	−1

[VTU 2003]

MODEL OBJECTIVE QUESTIONS

1. Two-person zero-sum game means that the
(a) sum of losses to one player equals the sum of gains to other.
(b) sum of losses to one player is not equal to the sum of gains to other.
(c) both (a) and (b). (d) none of the above.

2. Game theory models are classified by the
(a) number of players. (b) sum of all payoffs. (c) number of strategies. (d) all of the above.

3. A game is said to be fair, if
(a) both upper and lower values of the game are same and zero.
(b) upper and lower values of the game are not equal.
(c) upper value is more than lower value of the game.
(d) none of the above.

4. What happens when maximin and minimax values of the game are same?
(a) No solution exists. (b) Solution is mixed. (c) Saddle point exists. (d) None of the above.

5. A mixed strategy game can be solved by
(a) algebraic method. (b) matrix method. (c) graphical method. (d) all of the above.

6. The size of the payoff matrix of a game can be reduced by using the principle of
(a) game inversion. (b) rotation reduction. (c) dominance. (d) game transpose.

7. The payoff value for which each player in a game always selects the same strategy is called the
(a) saddle point. (b) equilibrium point. (c) both (a) and (b). (d) none of the above.

8. Games which involve more than two players are called
(a) conflicting games. (b) negotiable games. (c) n-person games. (d) all of the above.

9. When the sum of gains of one player is equal to the sum of losses to another player in a game, this situation is known as
(a) biased game. (b) zero-sum game. (c) fair game. (d) all of the above.

10. When no saddle point is found in a payoff matrix of a game, the value of the game is then found by
(a) knowing joint probabilities of each row and column combination to calculate expected payoff for that combination and adding all such values.
(b) reducing size of the game to apply algebraic method.
(c) both (a) and (b). (d) none of the above.

Answers

1. (c) **2.** (d) **3.** (a) **4.** (c) **5.** (d) **6.** (c) **7.** (c) **8.** (c) **9.** (b)
10. (c)

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10

QUEUEING SYSTEMS

(Waiting Line Models)

10.1. INTRODUCTION

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the ***waiting line,*** in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait, (*i.e.* remains idle), until additional people arrive. Here the arriving people are called the ***customers*** and the person issuing the tickets is called a ***server.***

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repairman. These examples show that the term *customer* may be interpreted in various number of ways. It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aeroplanes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations (*e.g.* at check out counters of supermarket).

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single queue as shown in *Fig.* 10.1 or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

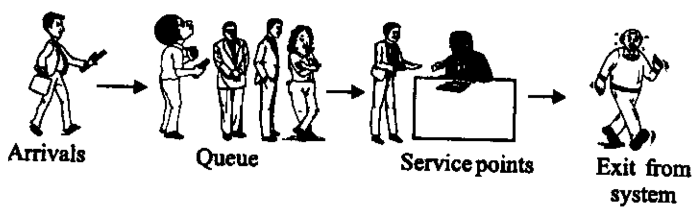

Fig. 10.1 (a). Queueing system with single queue and single service station.

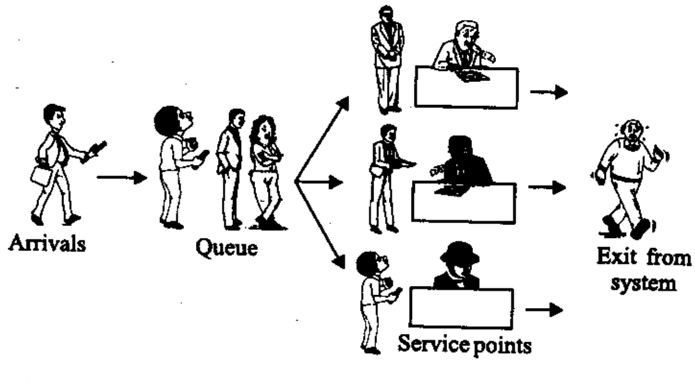

Fig. 10.1 (b). Queueing system with single queue and several service stations.